

Foundations of Numerical Reasoning in Grades K-2

Module 2: Building Fluency through 20

www.gfletchy.com



@gfletchy



gfletchy@gmail.com



Foundations of Numerical Reasoning in Grades K-2

Module 2: Building Fluency through 20

www.gfletchy.com



@gfletchy



gfletchy@gmail.com



Module Breakdown

Week 1: Early Number and Counting

Week 2: Building Fluency through 20

Week 3: Building Fluency through 120

Week 4: Building Fluency through 1000

Week 5: Number and Measurement

Week 6: Number and Statistical Reasoning



Module 2

Part 1: Launch the Concept

Part 2: Connect and Extend Thinking

Part 3: Building Fluency



3-Act Task



Notice? Wonder?

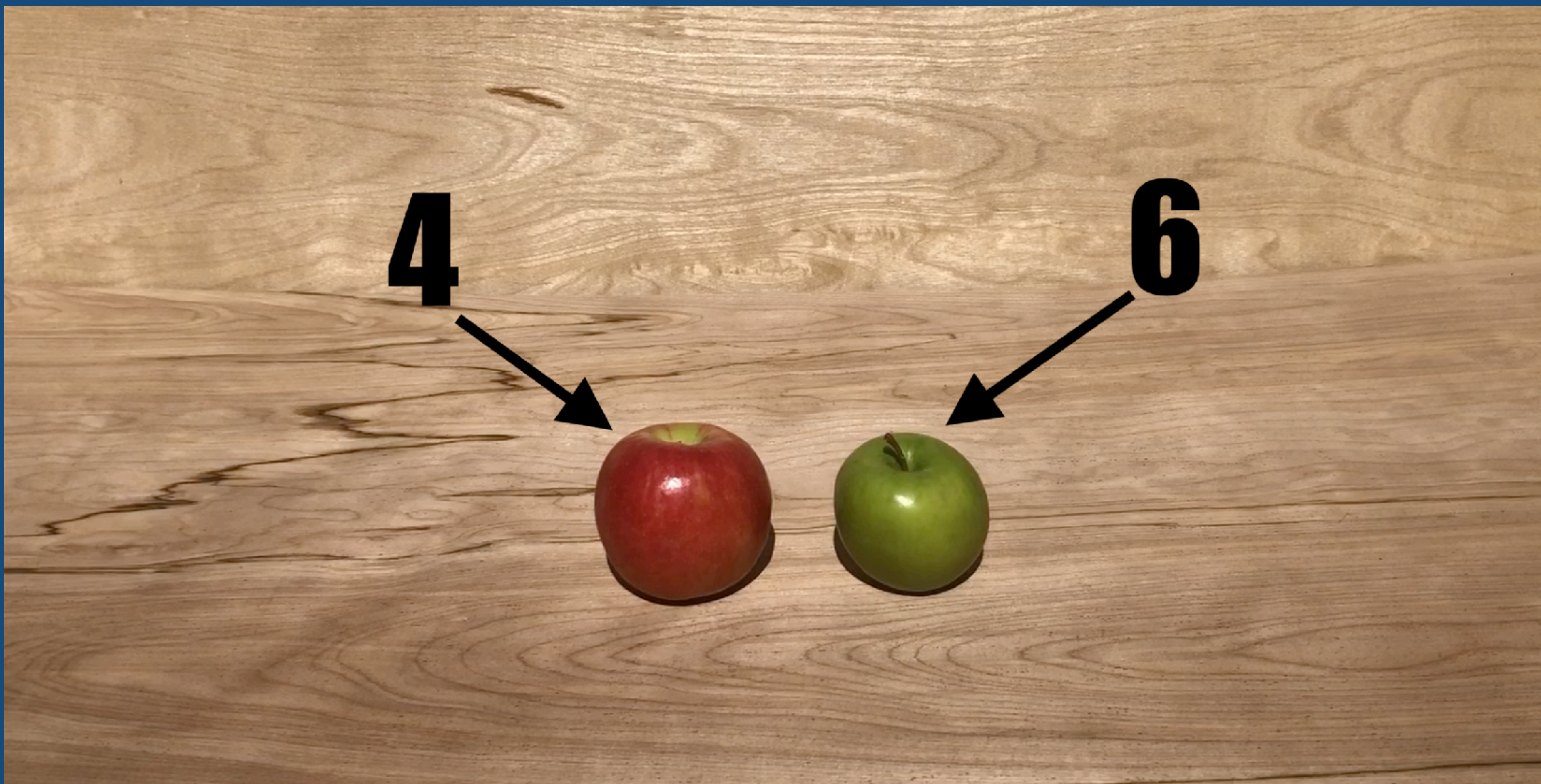


How many apples are in the basket?



What information would you like to know?



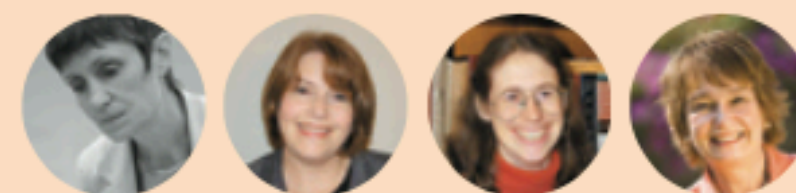




Orchestrating Discussions

Five practices constitute a model for effectively using student responses in whole-class discussions that can potentially make teaching with high-level tasks more manageable for teachers.

Margaret S. Smith, Elizabeth K. Hughes, Randi A. Engle, and Mary Kay Stein



Margaret S. Smith, pegs@pitt.edu, is an associate professor of mathematics education at the University of Pittsburgh. Over the past decade, she has been developing research-based materials for use in the professional development of mathematics teachers and studying what teachers learn from the professional development in which they engage. **Elizabeth K. Hughes**, elizabeth.hughes@uni.edu, recently finished her doctorate in mathematics education at the University of Pittsburgh. Her areas of interest include preservice secondary mathematics teacher education and the use of practice-based materials in developing teachers' understanding of what it means to teach and learn mathematics. **Randi A. Engle**, raengle@berkeley.edu, is an assistant professor of mathematics education and the social context of learning at the University of California Berkeley. She is interested in developing practical theories for how mathematics teachers can create discussion-based learning environments that promote strong student engagement, learning, and transfer. **Mary Kay Stein**, mkstei@pitt.edu, is a professor of learning sciences and policy and the director of the Learning Policy Center at the University of Pittsburgh. Her research focuses on instructional practice and the organizational and policy conditions that shape it.

Discussions that focus on cognitively challenging mathematical tasks, namely, those that promote thinking, reasoning, and problem solving, are a primary mechanism for promoting conceptual understanding of mathematics (Hatano and Inagaki 1991; Michaels, O'Connor, and Resnick forthcoming). Such discussions give students opportunities to share ideas and clarify understandings, develop convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives (NCTM 2000).

Although discussions about high-level tasks provide important

The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

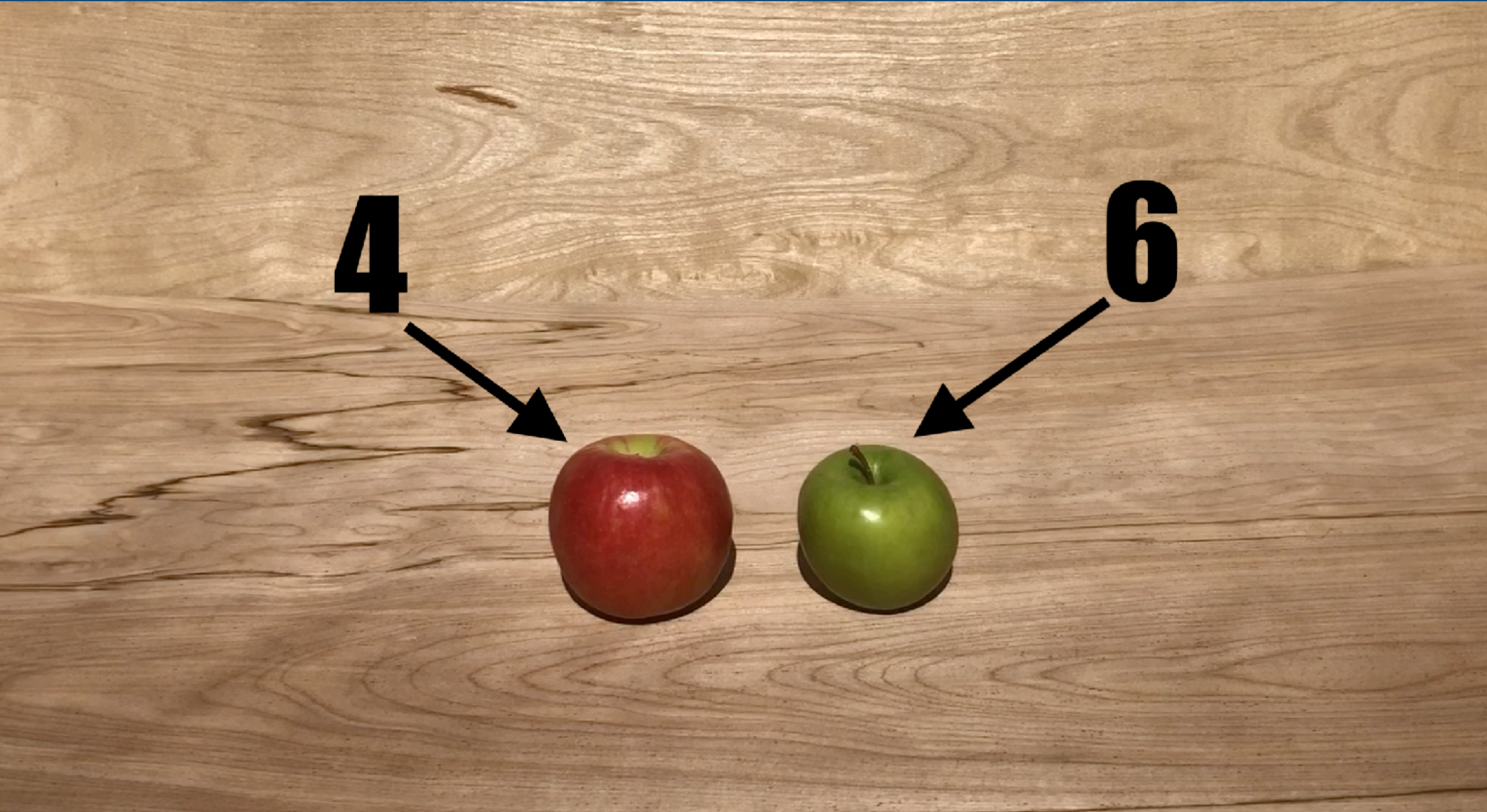
The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

Task Planning Page

Learning Target:		
Questions, Look-Fors and Misconceptions:		
Strategy	Who and What	Order
Scaffolding	Extension	Notes

Anticipating → Monitoring → Selecting → Sequencing → Connecting



Pause
II



The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

#2-Show your thinking

Student 1

$$4+4=8+2=10$$

"I know $4+4=8$ but I need 2 more green. I pictured a ten-frame in my brain"

#2-Show your thinking

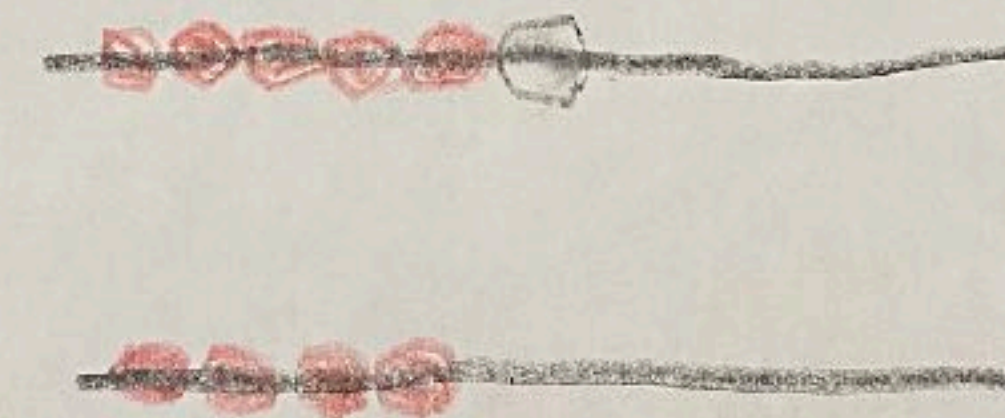
Student 2



Unable to represent with a drawing so we modeled with red and green unifix cubes. Counted a total of 9 cubes.

#2-Show your thinking

Student 3



"I used my rekenrek. I made the top row red and the bottom row green. Then I counted them all and I got ten."

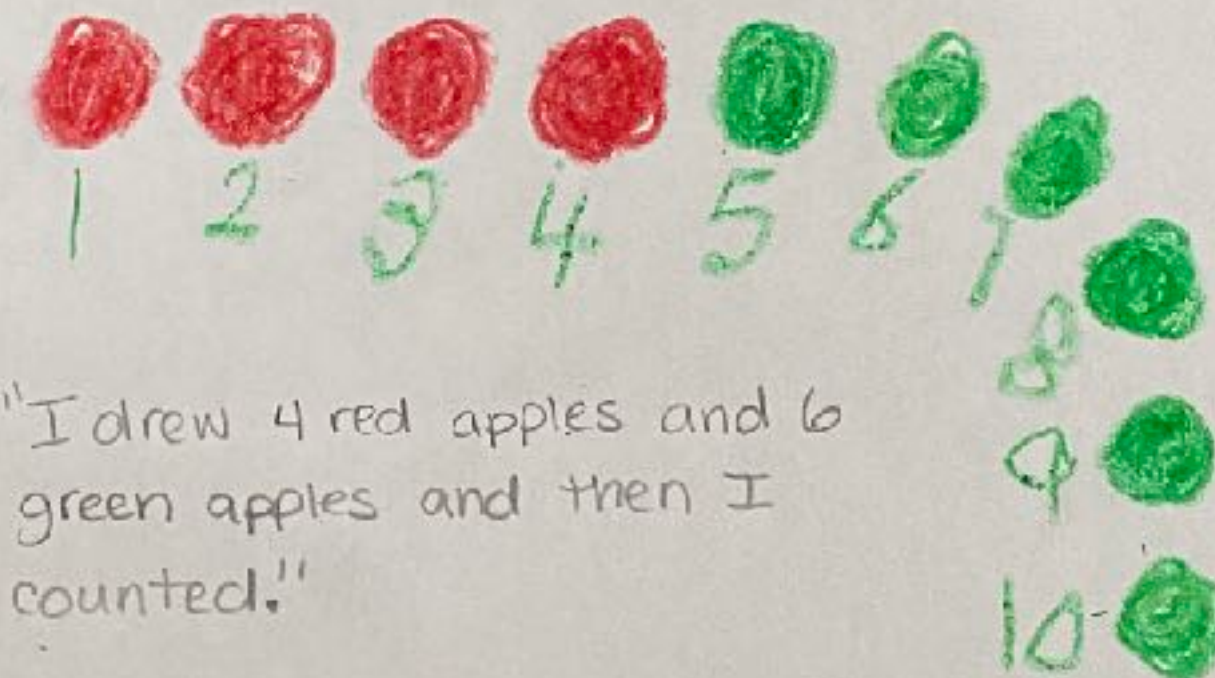
#2-Show your thinking

$$6+4=10$$

"I just know that one. It's just like $5+5$ but I turned a red apple into green."

Student 4

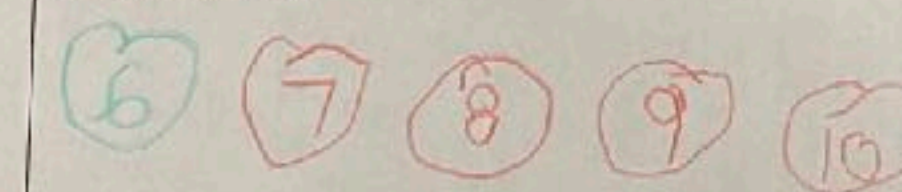
#2-Show your thinking



"I drew 4 red apples and 6 green apples and then I counted."

Student 5

#2-Show your thinking



"I counted on four more from six and I got to ten."

Student 6

Pause

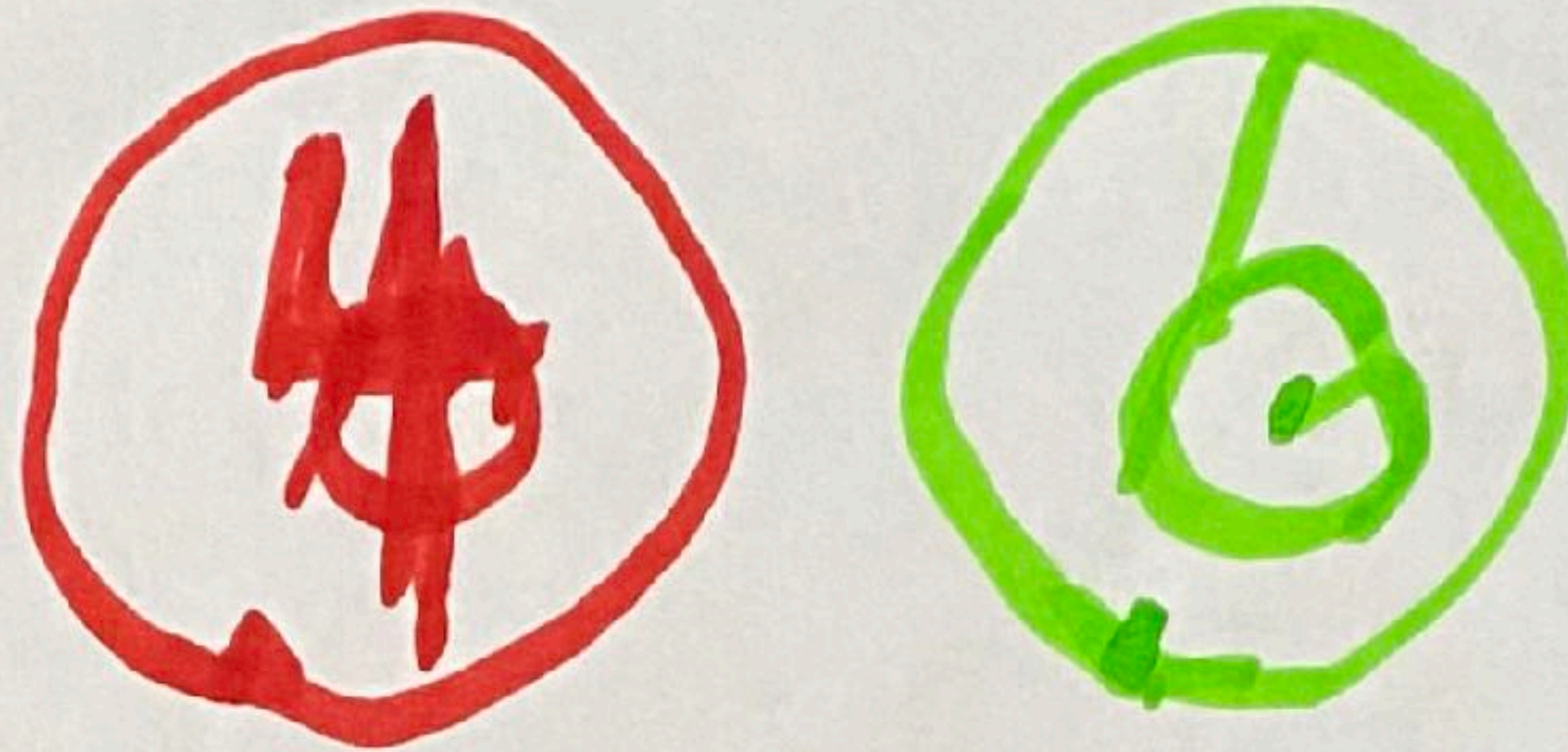


#2-Show your thinking

Student 1

$$4 + 4 = 8 + 2 = 10$$

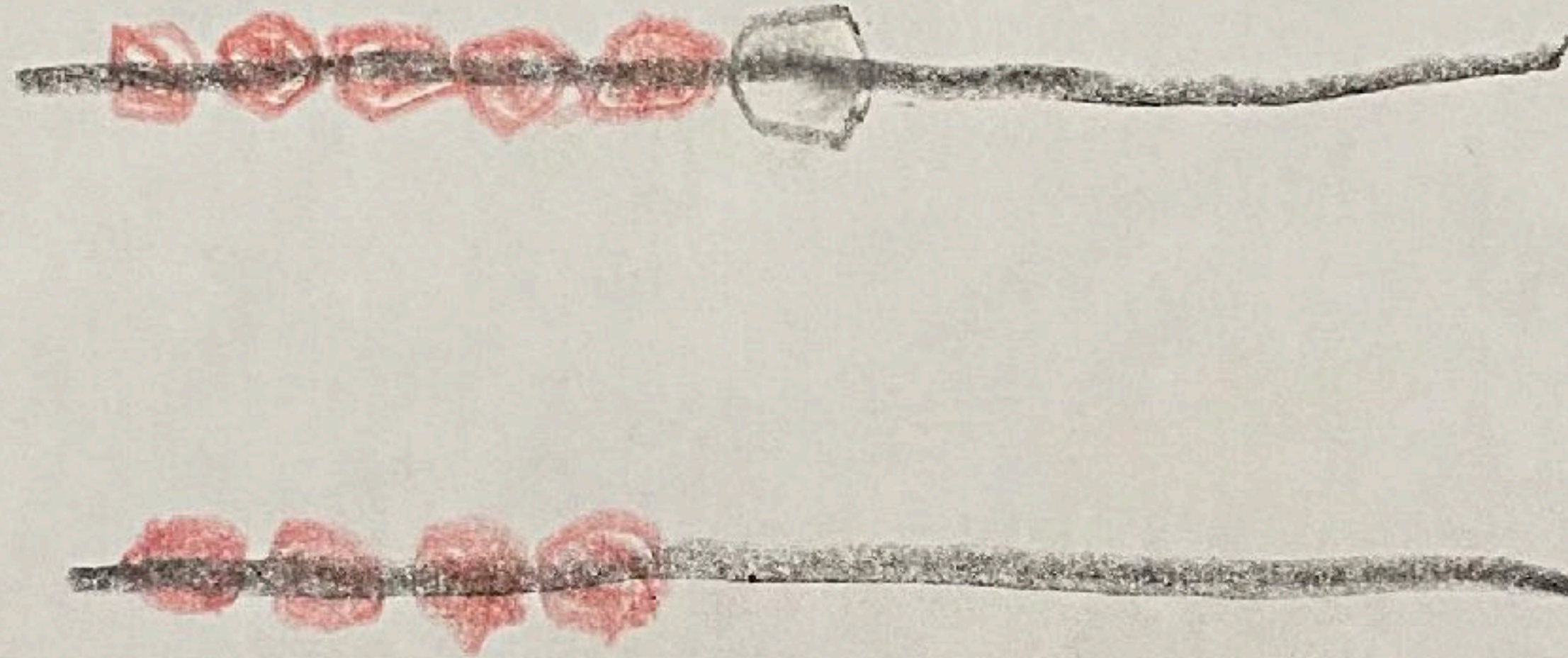
"I know $4 + 4 = 8$ but I need 2 more green. I pictured a ten-frame in my brain



Unable to represent with a drawing so we modeled with red and green unifix cubes. Counted a total of 9 cubes.

#2-Show your thinking

Student 3



"I used my rekenrek. I made the top row red and the bottom row green. Then I counted them all and I got ten."

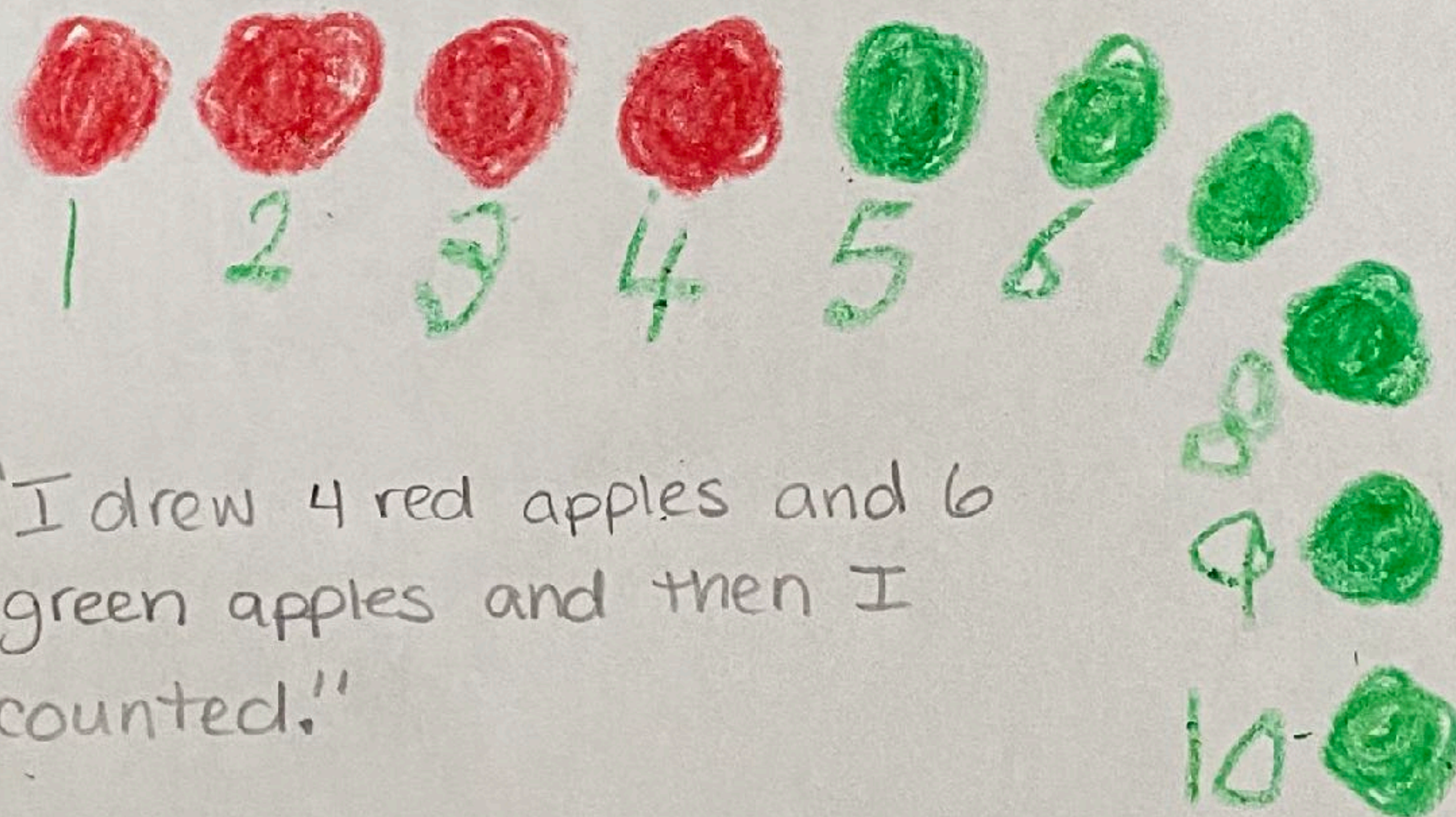
#2-Show your thinking

$$6 + 4 = 10$$

"I just know that one. It's just like $5+5$ but I turned a red apple into green."

Student 4

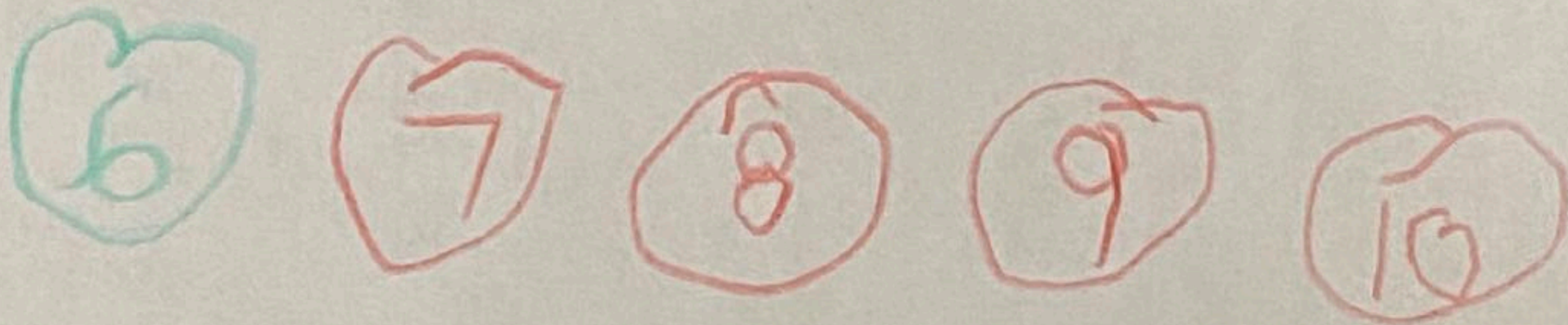
#2-Show your thinking



"I drew 4 red apples and 6
green apples and then I
counted."

Student 5

#2-Show your thinking



"I counted on four more
from six and I got to ten."

Student 6

The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

#2-Show your thinking

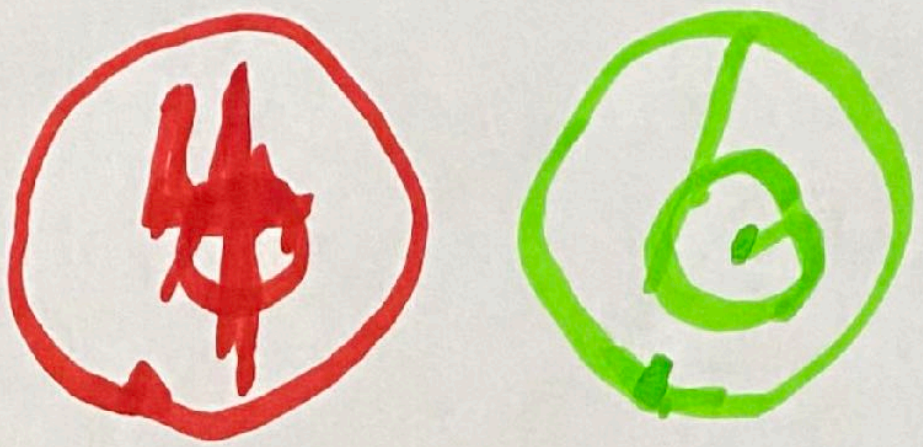
Student 1

$$4+4=8+2=10$$

"I know $4+4=8$ but I need 2 more green. I pictured a ten-frame in my brain"

#2-Show your thinking

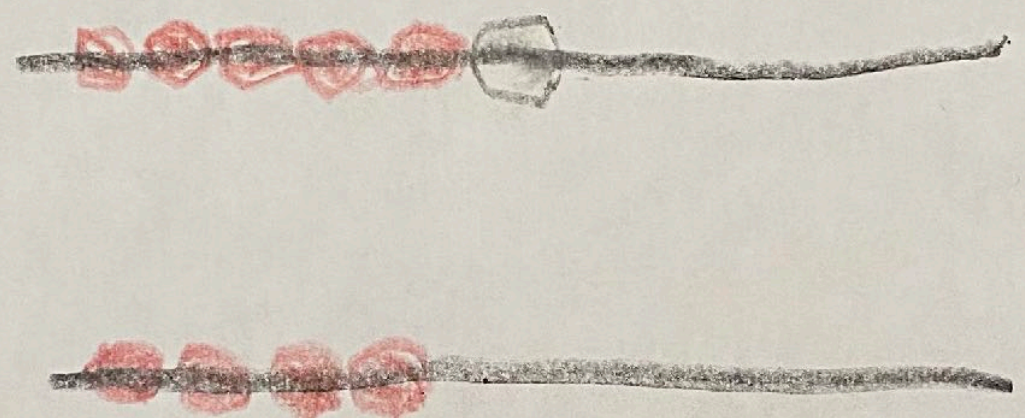
Student 2



Unable to represent with a drawing so we modeled with red and green unifix cubes. Counted a total of 9 cubes.

#2-Show your thinking

Student 3



"I used my rekenrek. I made the top row red and the bottom row green. Then I counted them all and I got ten."

How would you sequence or connect these student work samples?

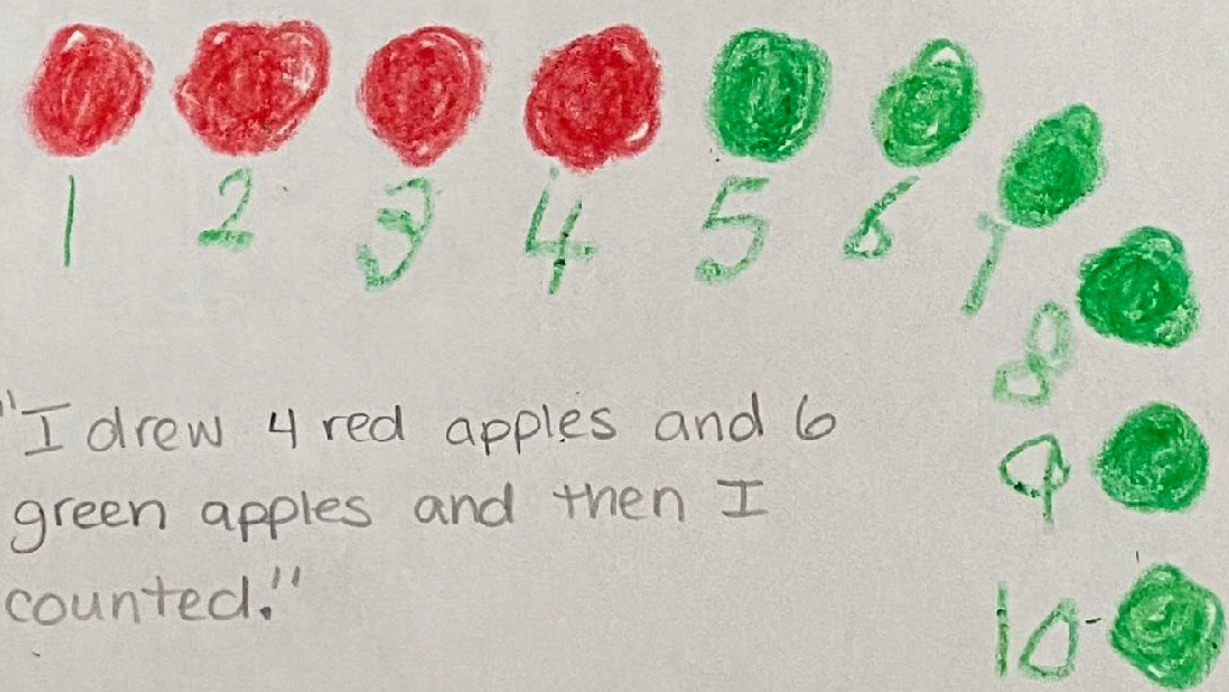
#2-Show your thinking

$$6+4=10$$

"I just know that one. It's just like $5+5$ but I turned a red apple into green."

Student 4

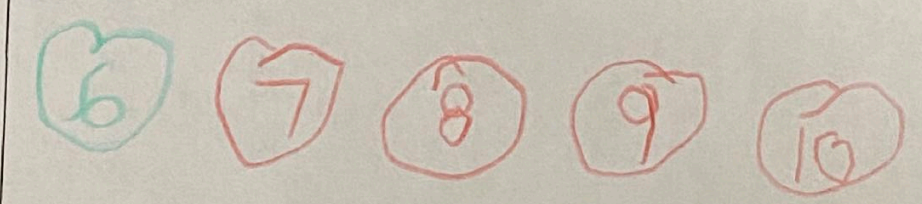
#2-Show your thinking



"I drew 4 red apples and 6 green apples and then I counted."

Student 5

#2-Show your thinking

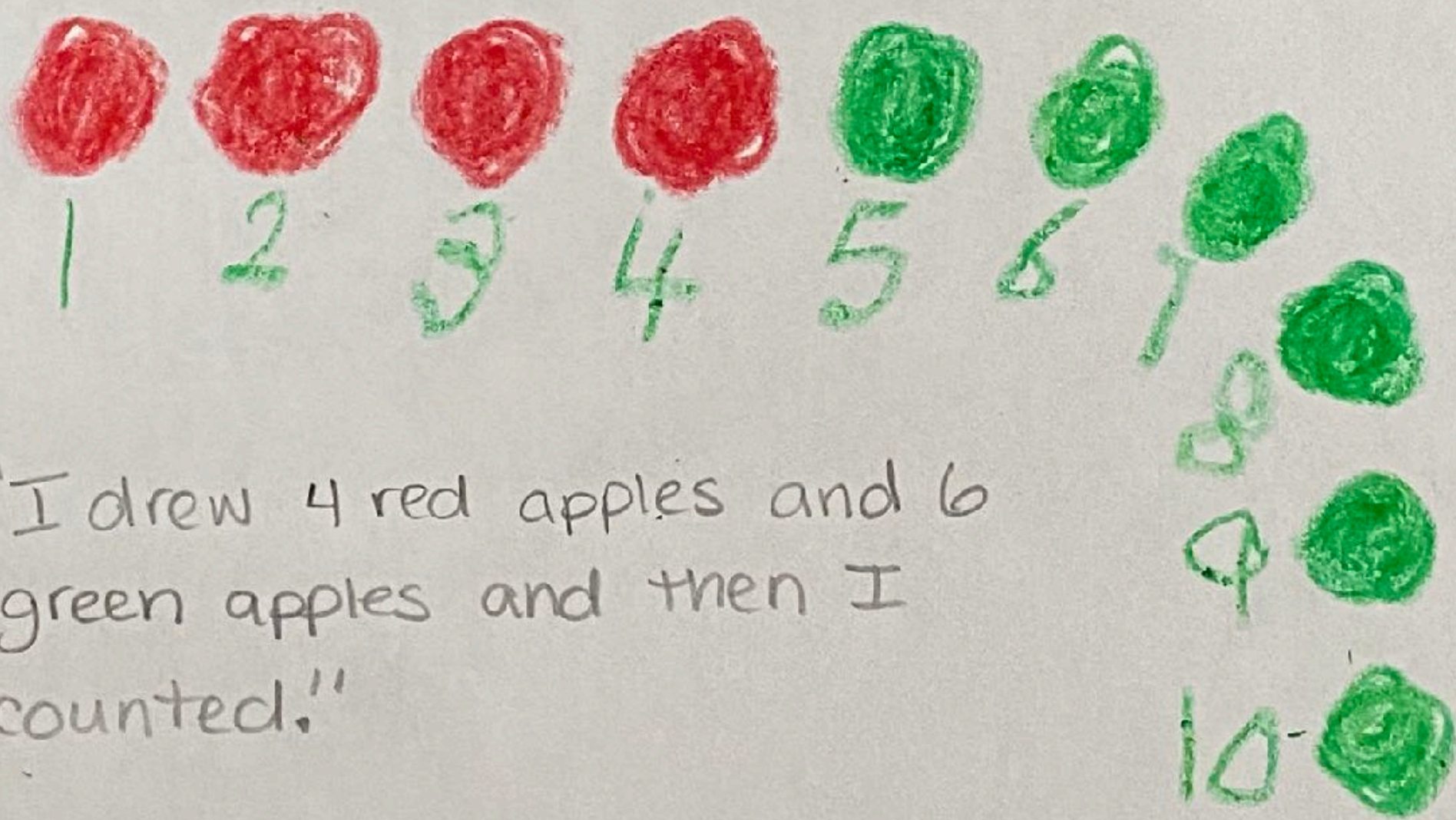


"I counted on four more from six and I got to ten."

Student 6

Sequence and Connect

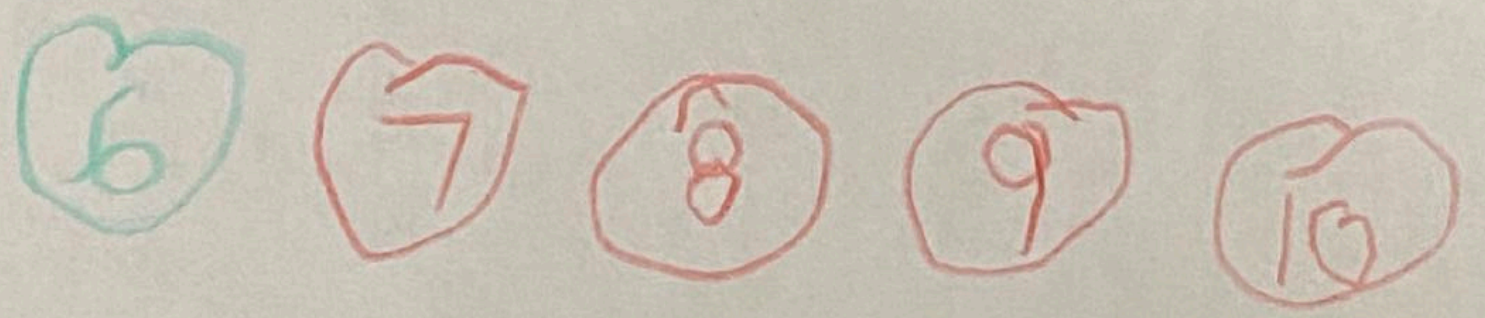
#2-Show your thinking



"I drew 4 red apples and 6 green apples and then I counted."

Student 5

#2-Show your thinking



"I counted on four more from six and I got to ten."

Student 6

Version #2

Comparison Problem



10 Apples



2 more green apples than red.

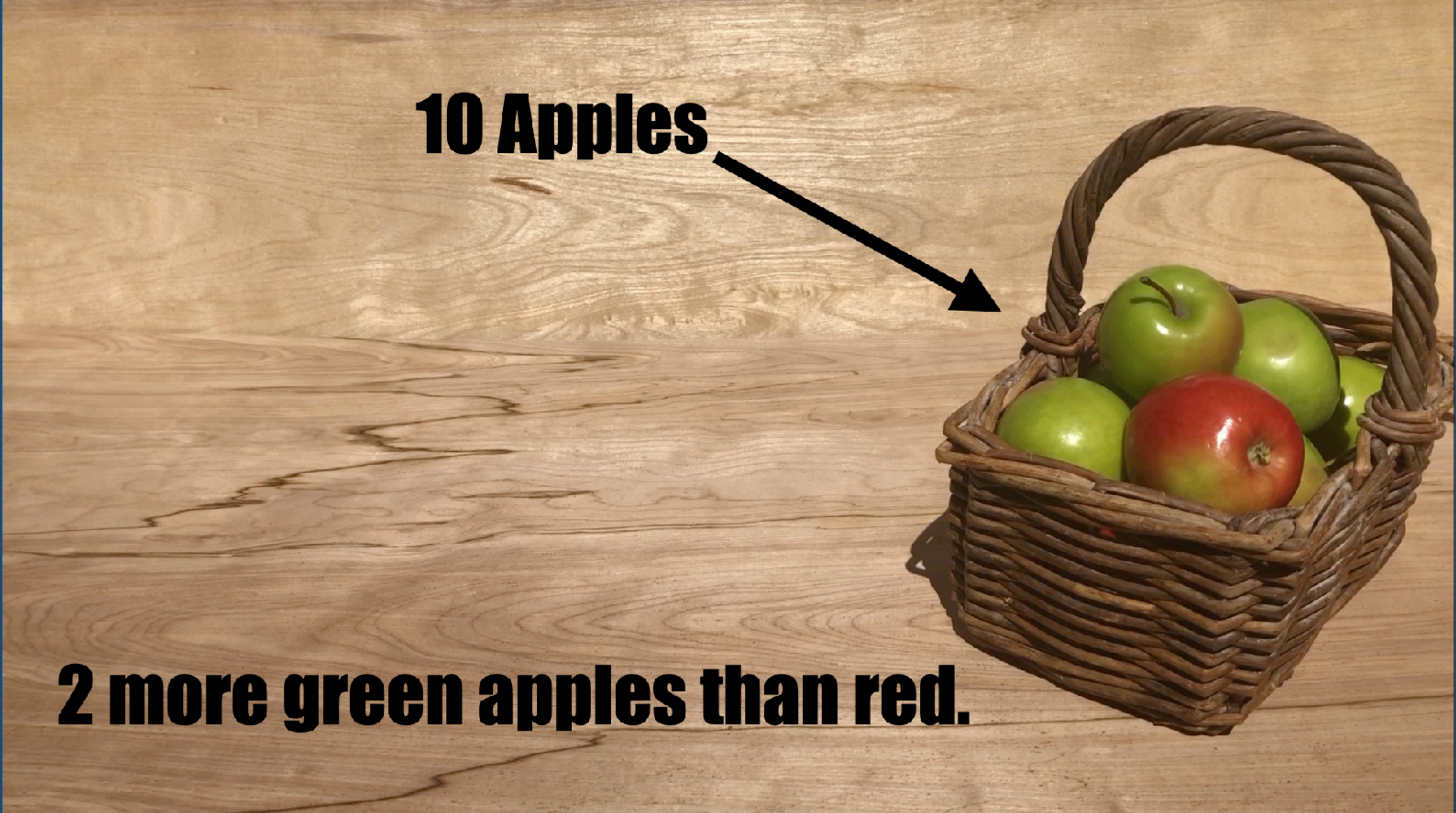
The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

Task Planning Page

Learning Target:		
Questions, Look-Fors and Misconceptions:		
Strategy	Who and What	Order
Scaffolding	Extension	Notes

Anticipating → Monitoring → Selecting → Sequencing → Connecting



Pause
||



The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

#2-Show your thinking

Student 1

#2-Show your thinking

Student 2

#2-Show your thinking

Student 3

#2-Show your thinking

Student 4

#2-Show your thinking

Student 5

#2-Show your thinking

Student 6

Pause



#2-Show your thinking

Student 1



1
2
3
4



3
4
5
6



#2-Show your thinking

Student 2

$$\cancel{1+9=10}$$

$$\cancel{8+2=10}$$

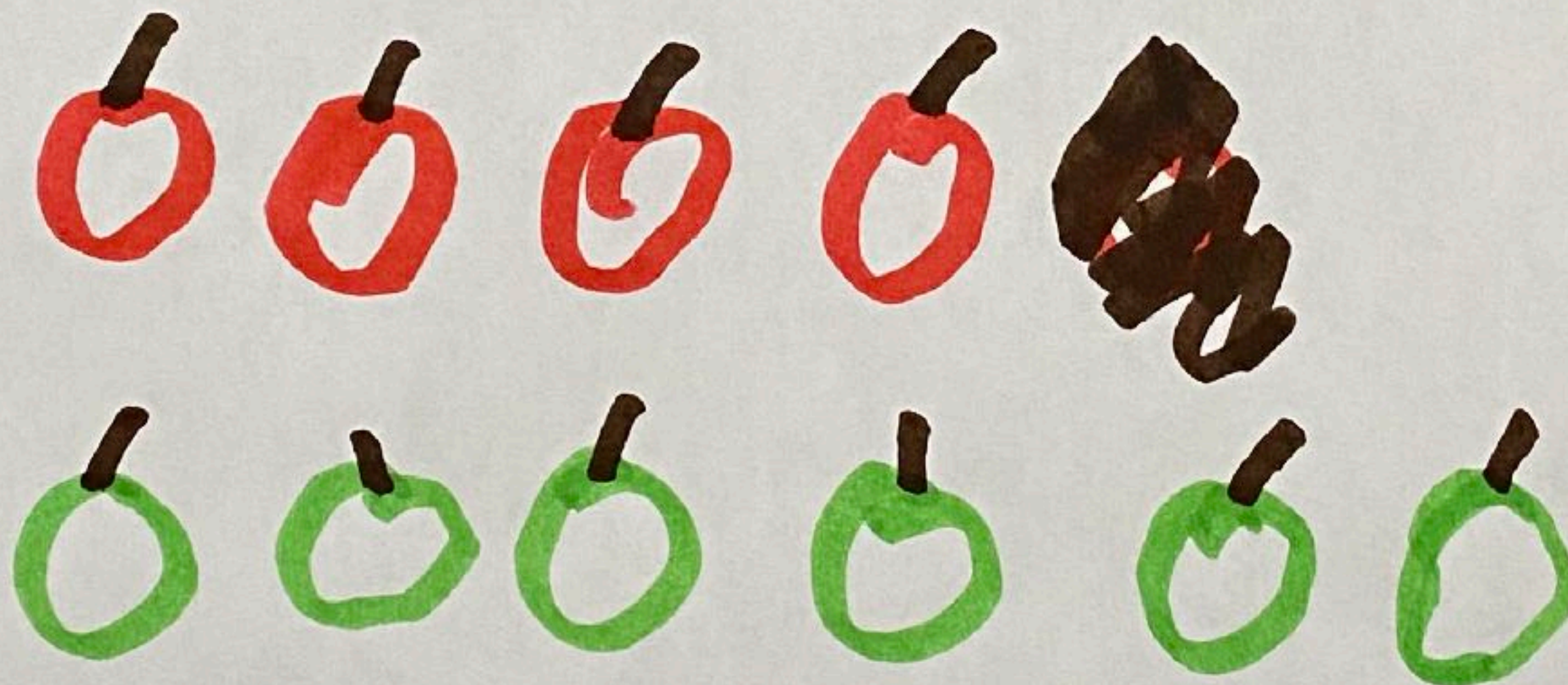
$$\cancel{7+3=10}$$

$$6+4=10$$

$$\cancel{5+5=10}$$

#2-Show your thinking

Student 3



10

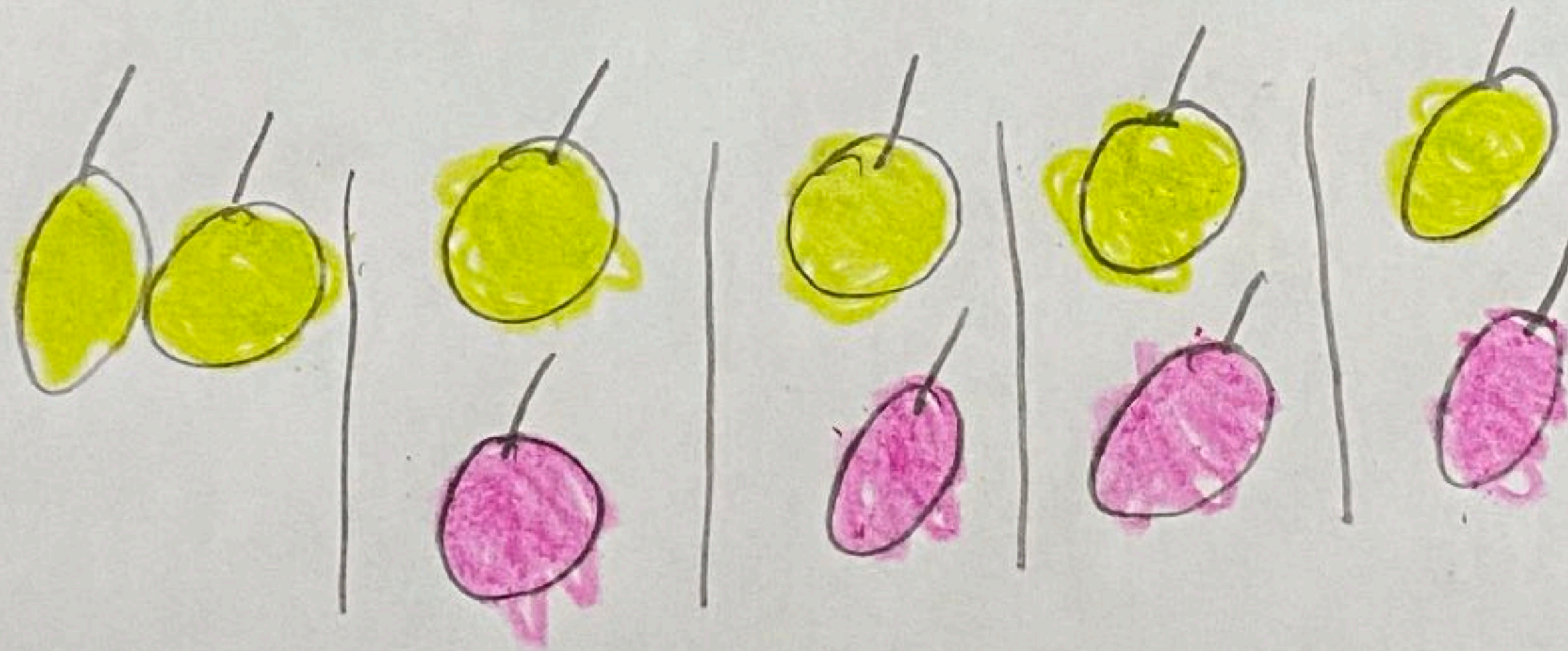
#2-Show your thinking

9	9	9	9	9
1	1	1	1	9

$$6 + 4 = 10$$

Student 4

#2-Show your thinking



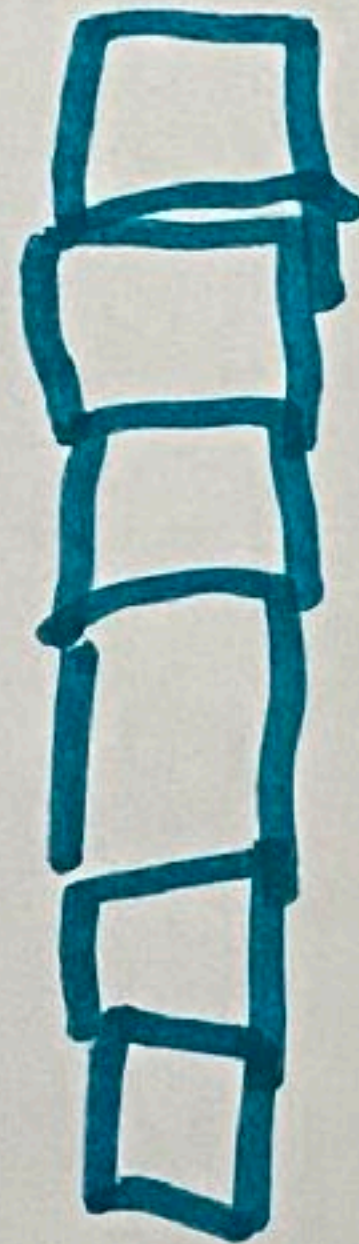
$$6 + 4 = 10$$

Student 5

#2-Show your thinking



4



6

Student 6

The **5** Practices:

1. **Anticipating** student responses to challenging mathematical tasks;

3. **Monitoring** students' work on and engagement with the tasks;

5. **Selecting** particular students to present their mathematical work;

6. **Sequencing** the student responses that will be displayed in a specific order and;

7. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

#2-Show your thinking

Student 1

1
2
3
4

3
4
5
6

★

#2-Show your thinking

Student 2

~~1+9=10~~
~~8+2=10~~
~~7+3=10~~
~~6+4=10~~
~~5+5=10~~

#2-Show your thinking

Student 3

10

How would you sequence or connect these student work samples?

#2-Show your thinking

6 + 4 = 10

Student 4

#2-Show your thinking

6 + 4 = 10

Student 5

#2-Show your thinking

4

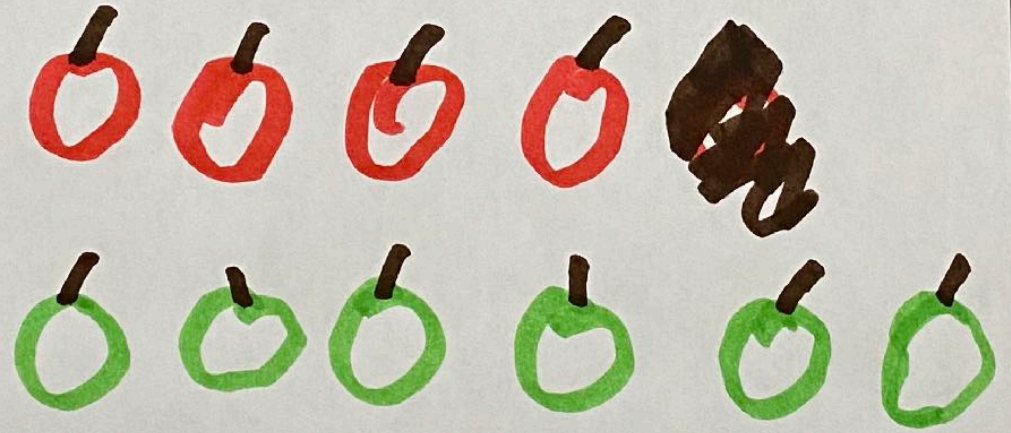
6

Student 6

Sequence and Connect

#2-Show your thinking

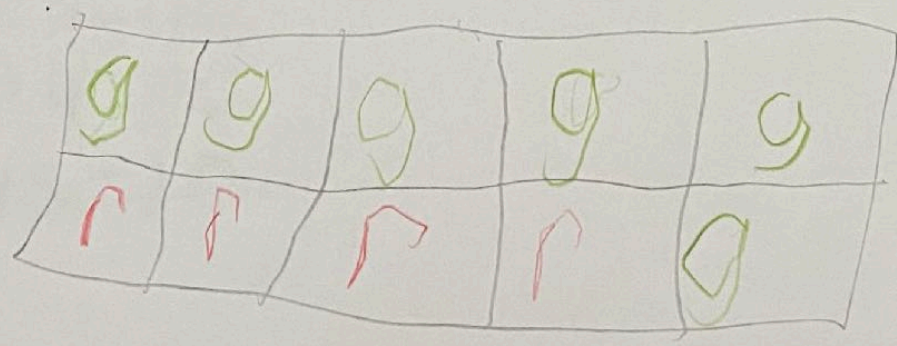
Student 3



10

Student 3's work shows a drawing of 10 apples arranged in two rows. The top row has 5 apples, with the first 4 being red and the last one being brown. The bottom row has 6 green apples. Below the drawing, the number 10 is written in black ink.

#2-Show your thinking



$6 + 4 = 10$

Student 4

Student 4's work shows a ten-frame with 10 dots. The top row has 5 green dots and the bottom row has 5 red dots. Below the ten-frame, the equation $6 + 4 = 10$ is written in green and red ink. The name Student 4 is written in black ink at the bottom.

#2-Show your thinking

Student 2

$1+9=10$
 $8+2=10$
 $7+3=10$
 $6+4=10$
 $5+5=10$

Student 2's work shows five equations written in black ink: $1+9=10$, $8+2=10$, $7+3=10$, $6+4=10$, and $5+5=10$. The name Student 2 is written in black ink at the top right.

We're learning who is...

- Combining sets
- Counting all
- Counting on
- Part-whole strategies
- Using counting strategies vs additive strategies

Module 2

Part 1: Launch the Concept

Part 2: Connect and Extend Thinking

Part 3: Building Fluency



What is fluency?





Thoughts?

Pause
II

1. What is fluency?
2. How do you build and support fact fluency in your classroom? Ideas? Resources?
3. How do you assess fluency?
4. What are some barriers, frustrations, or pitfalls around fact fluency?

What research says...



Adding It Up:

Procedural fluency refers to the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

(National Research Council, 2001, 121)



Principles to Actions:

Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems.

(National Council of Teachers of Mathematics, 2014, 42)



Procedural Fluency in Mathematics:

Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another.

(A Position of the National Council of Teachers of Mathematics, 2014)



Accurate

Finding the right answer.

Efficient

Solving a problem with a strategy that saves time or labor.

Flexible

Knowing multiple solution paths and choosing an appropriate strategy for a particular problem.



Accurate

Efficient

Flexible



$$2 + 9$$

$$2 + 21$$

$$2 + 99$$



Fact fluency means students
know and can *use* their facts.



Adding It Up:

Procedural fluency refers to the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

(National Research Council, 2001, 121)



Principles to Actions:

Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems.

(National Council of Teachers of Mathematics, 2014, 42)



Procedural Fluency in Mathematics:

*Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to **different problems and contexts**; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another.*

(A Position of the National Council of Teachers of Mathematics, 2014)



What I got right.

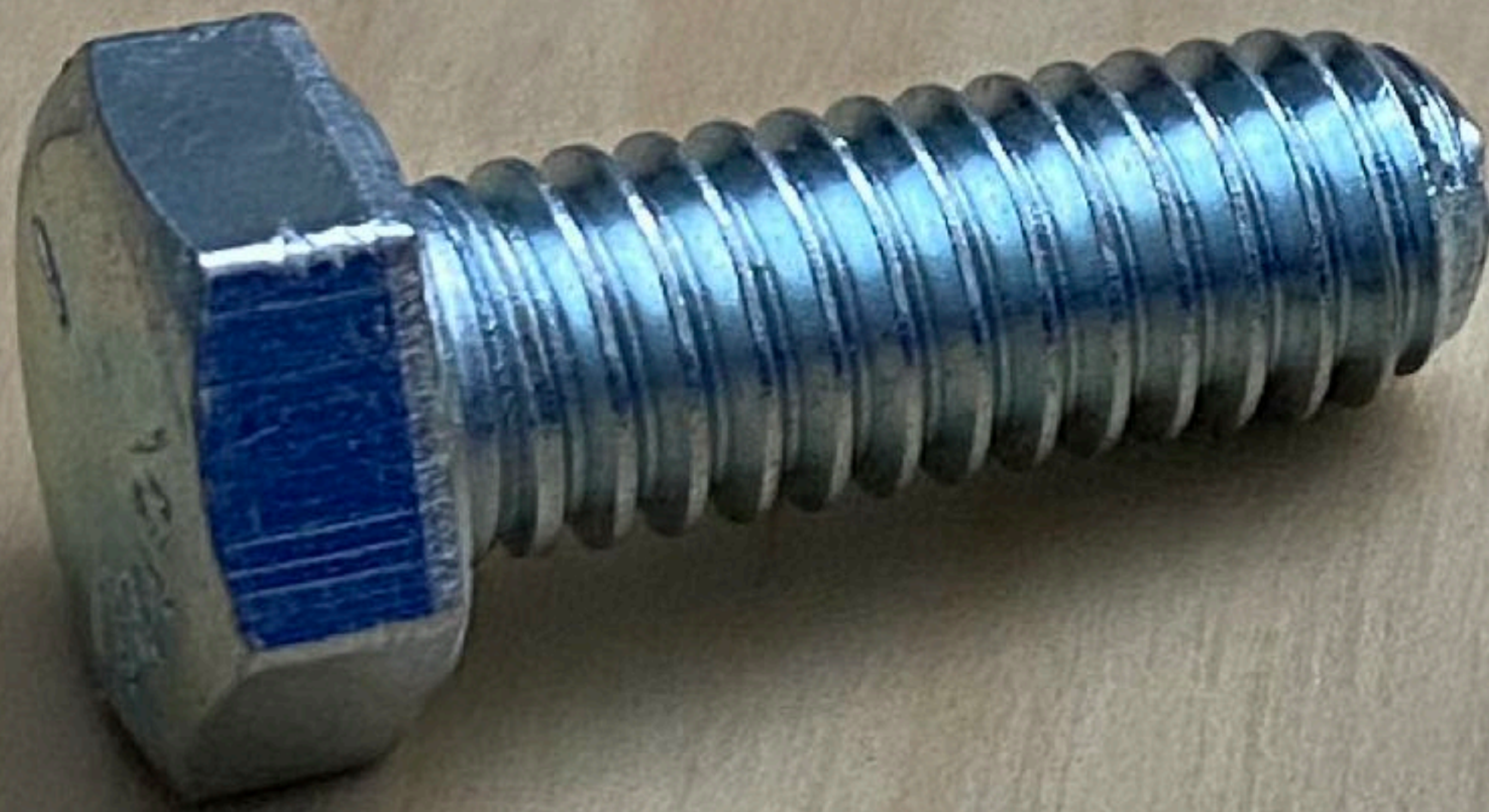
What I was missing.



Accuracy, Efficiency, Flexibility

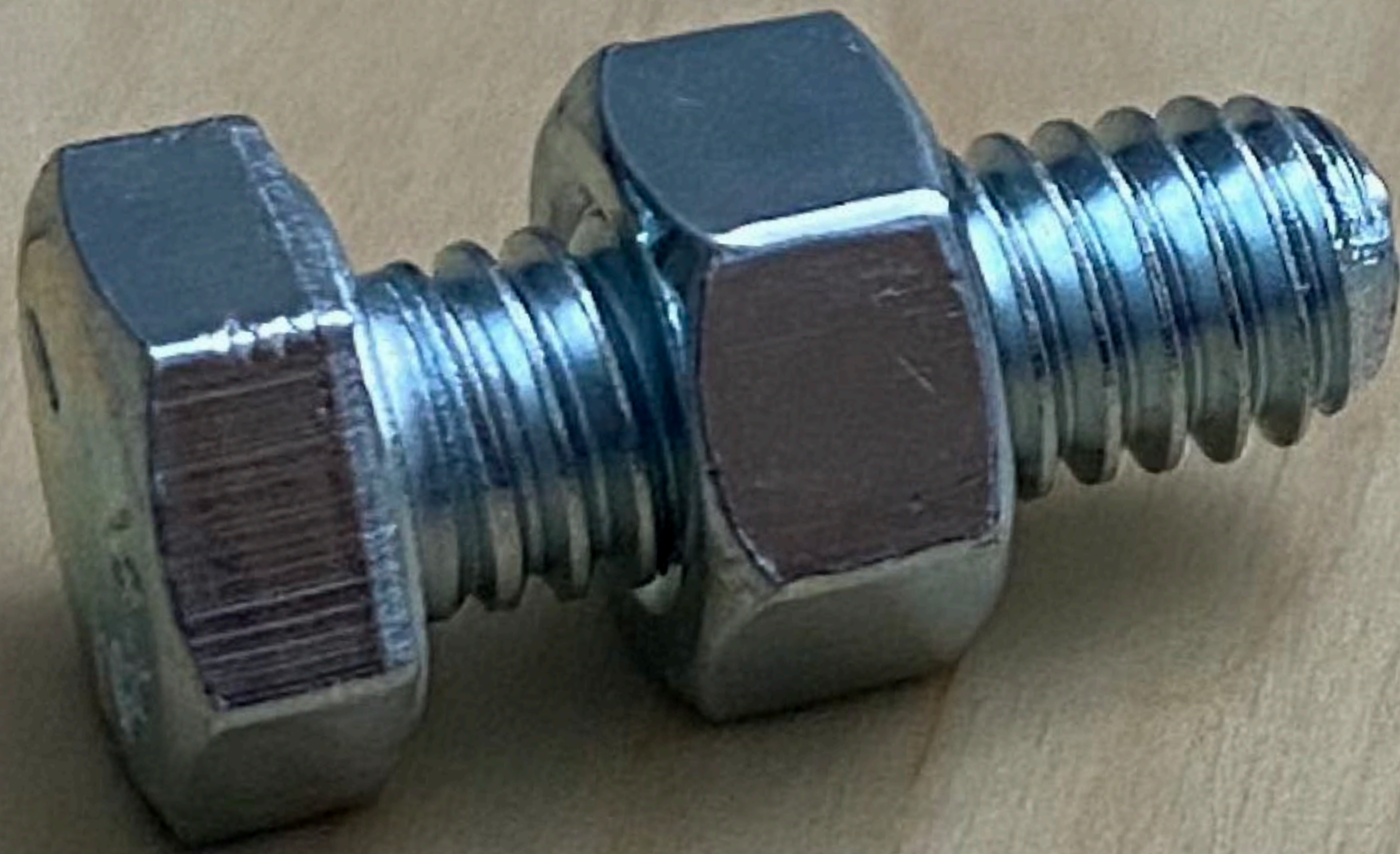


Accuracy, Efficiency, Flexibility



Context

The Nuts and Bolts of Fluency



Accuracy
Efficiency
Flexibility
Context

Lesson String

3 Act Task

Image Talk

Anchor Problem

Tool Talk

Contextualized Practice Problems

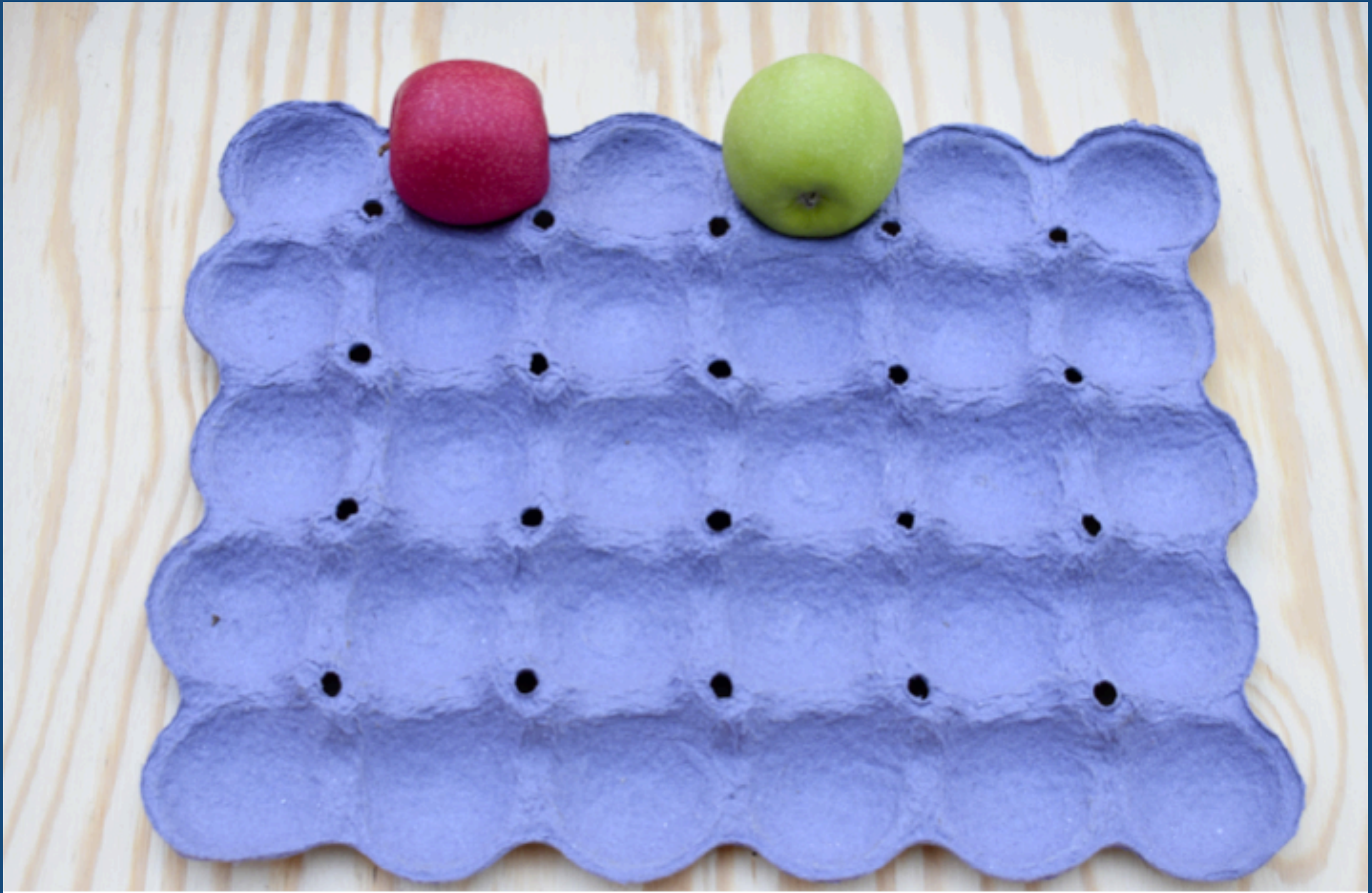
Game

Number Talk



Image Talk













The Problem with Word Problems



Anchor Problems



Felix had 4 apples. He had 2 red apples, and the rest were green apples. How many green apples did Felix have?

(8,4)

(14,7)

(26,13)



Felix has some apples. He has the same number of red apples and green apples. How many apples of each color could he have?



Felix had **4** apples. He had **2** red apples, and the rest were green apples. How many green apples did Felix have?

Same? Different?

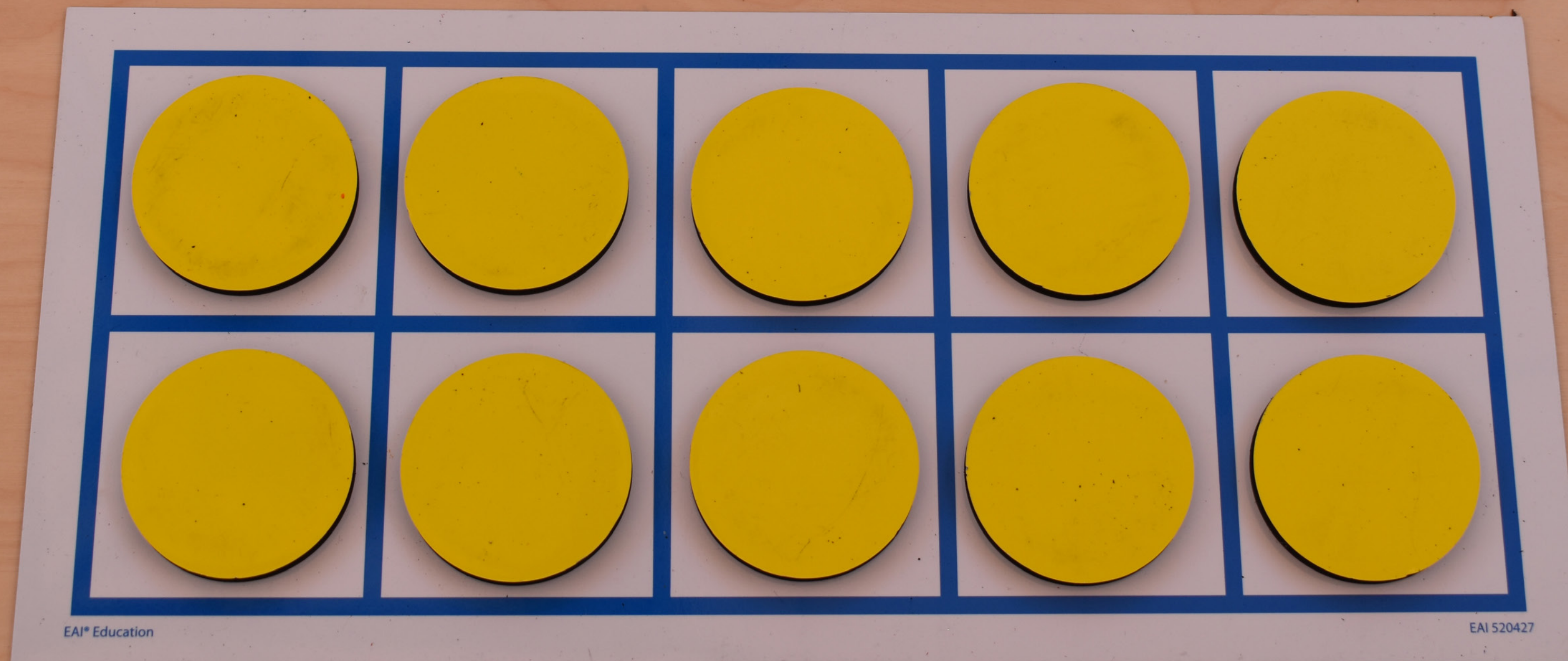
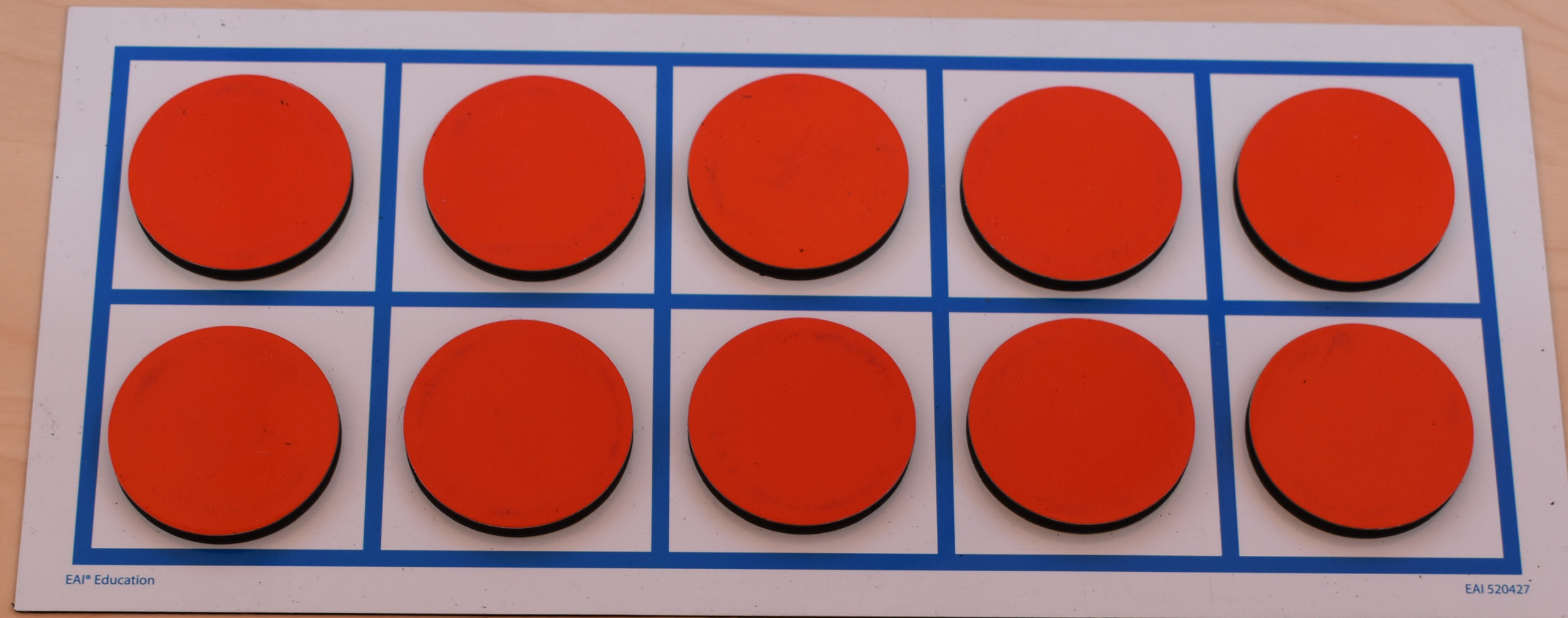
Felix has some apples. He has the same number of red apples and green apples. How many apples of each color could he have?

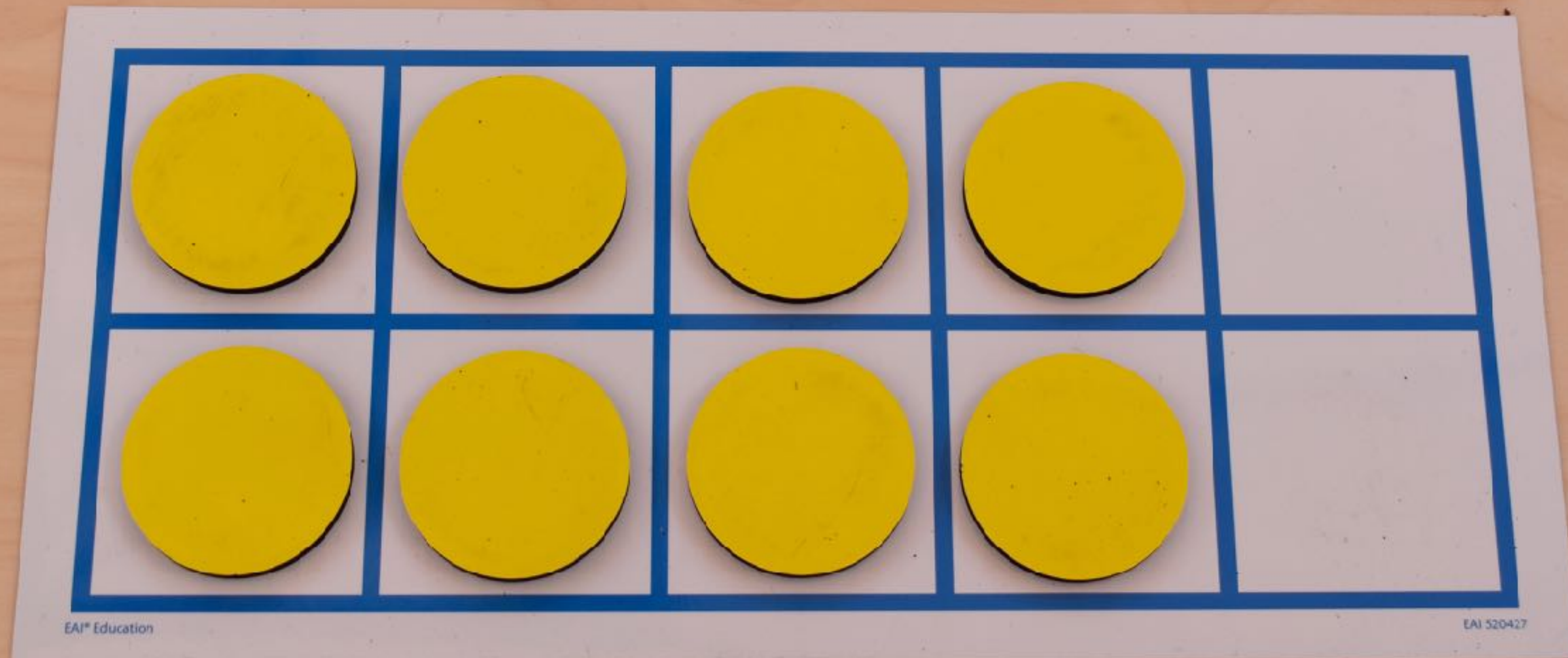
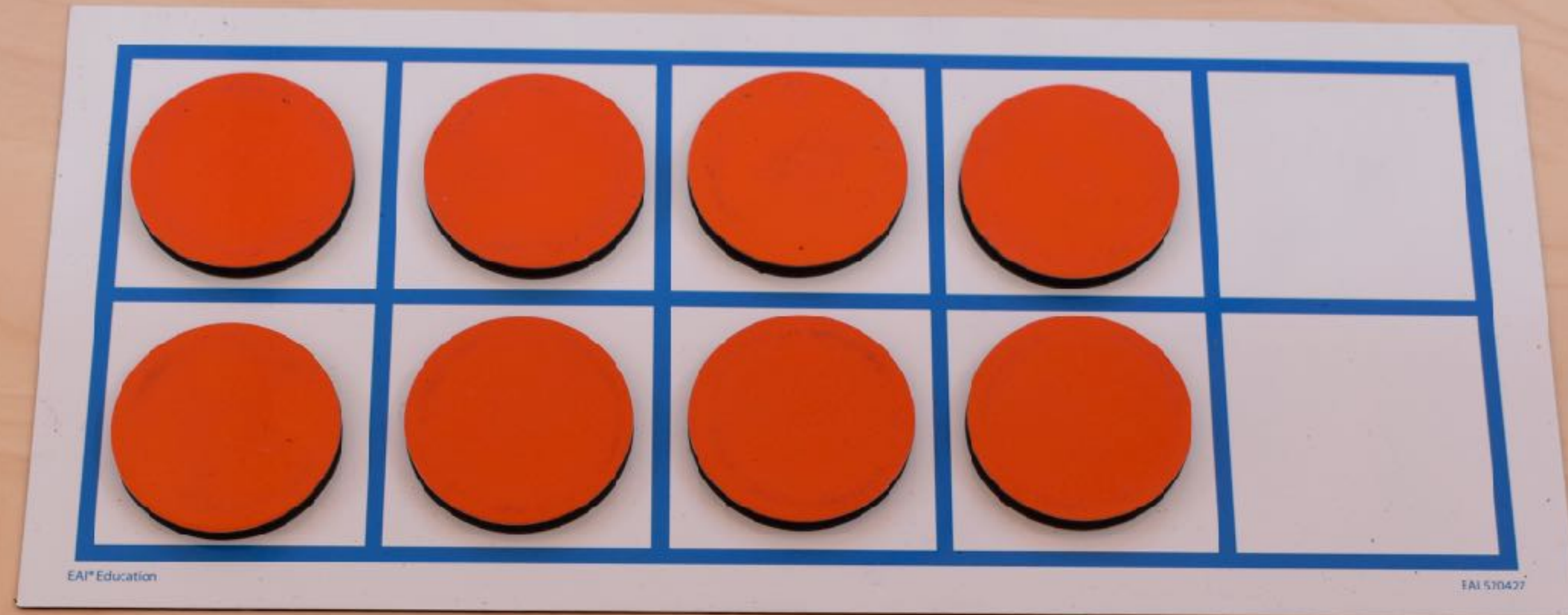
Decontextualizing & Recontextualizing Number

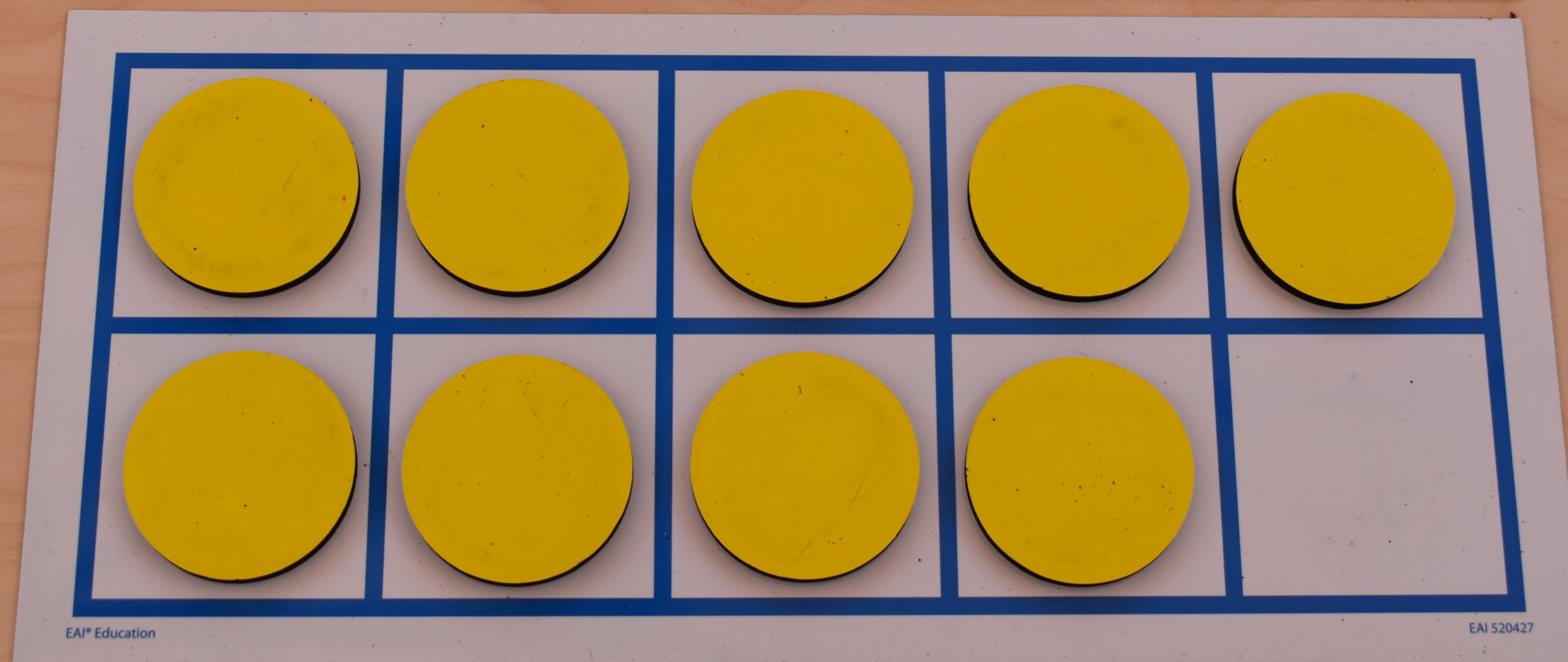
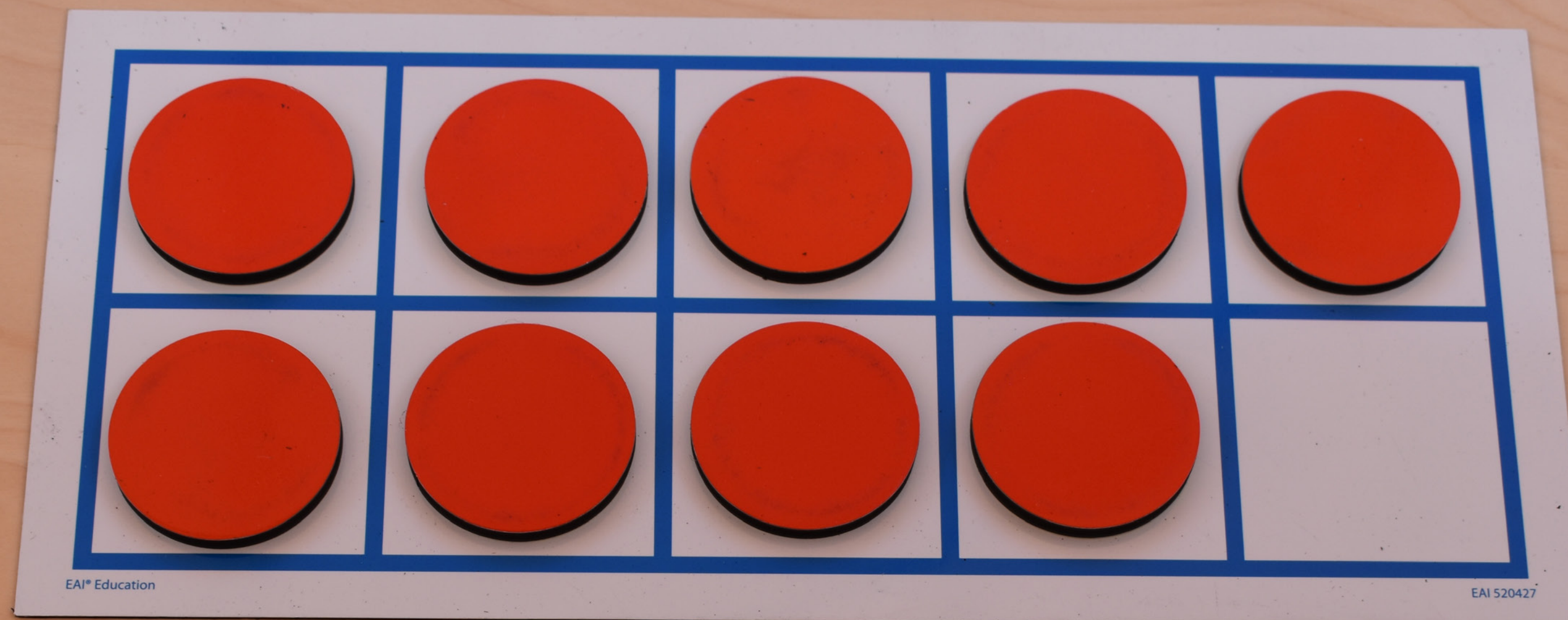


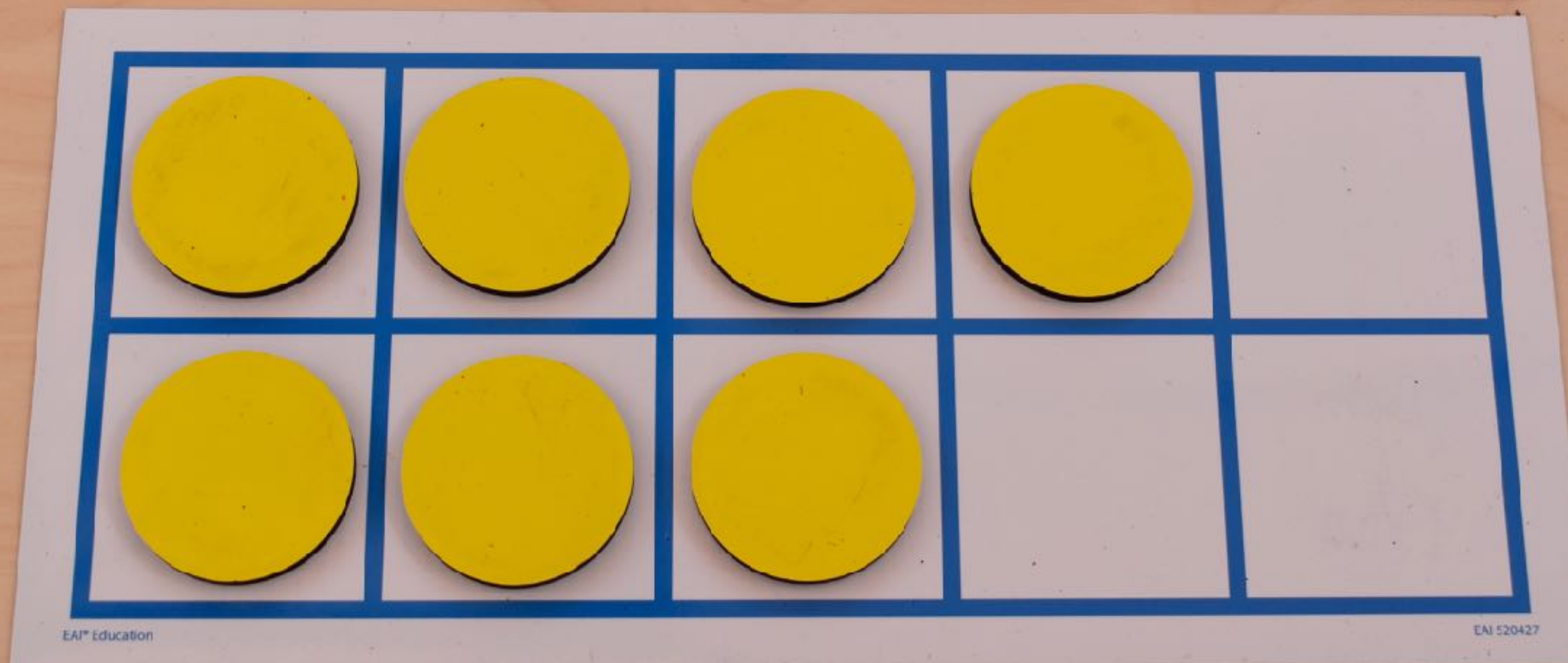
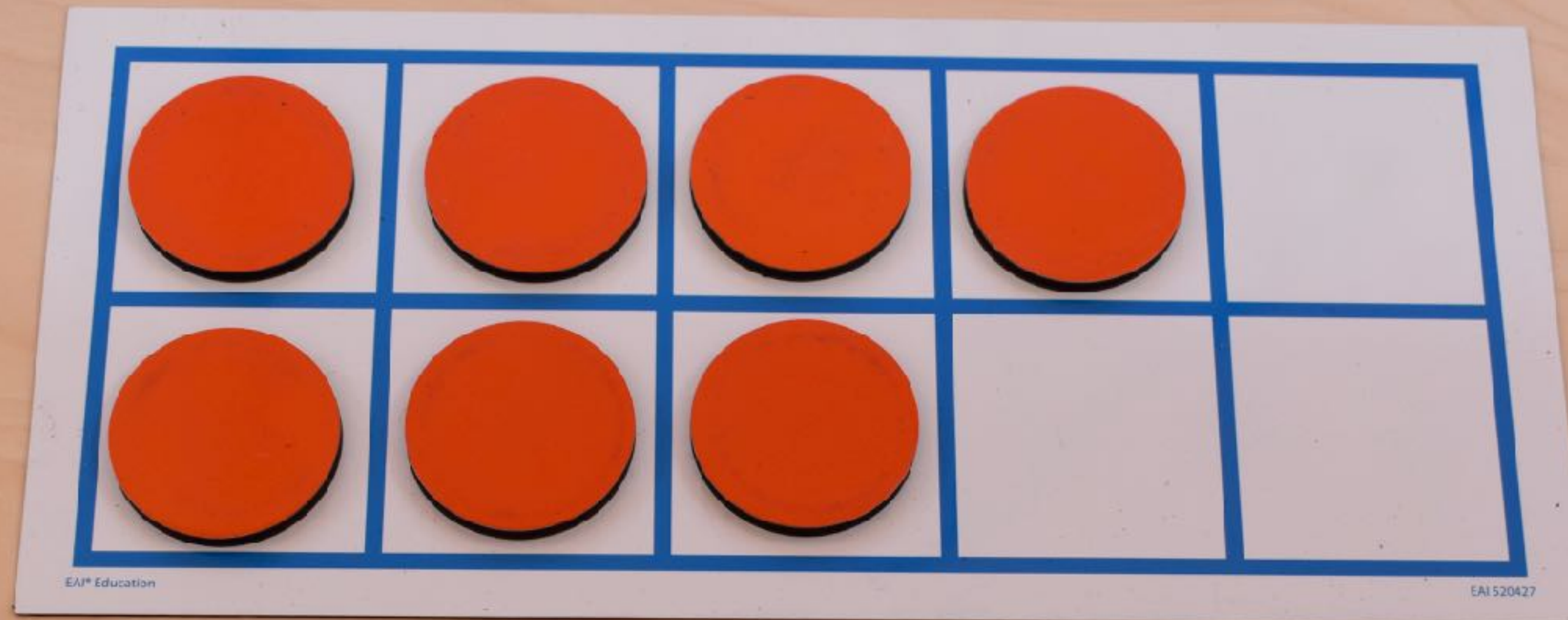
Tool Talk







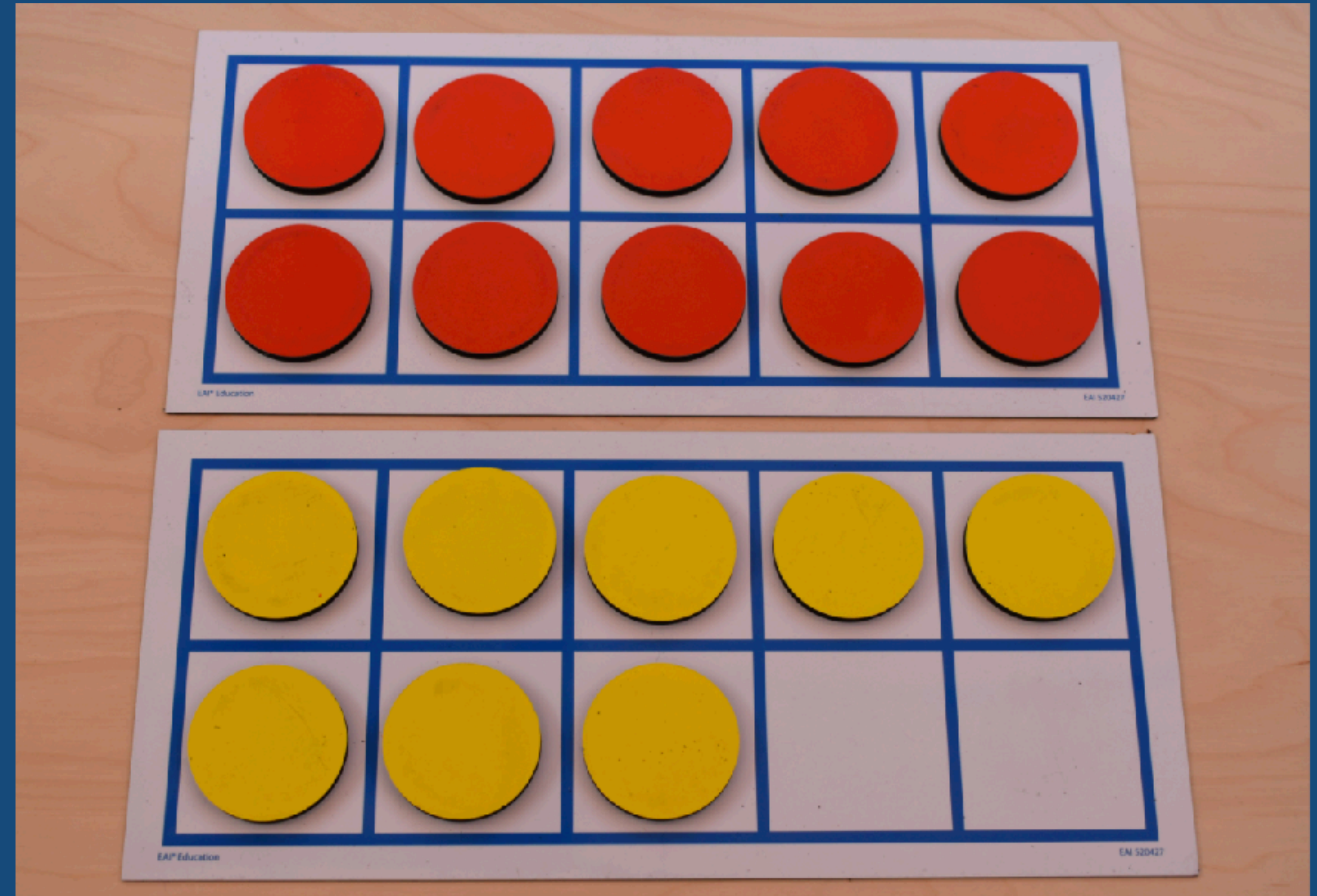




Decontextualizing & Recontextualizing Number



Same? Different?



Contextualized Practice Problems



Doubles—Contextualized Practice (Apples)



	Result Unknown	Change Unknown	Start Unknown
Join (Add To)	Felix had apples. Lauren gave him more apples. How many apples does Felix have now? [4, 4] [7, 7] [11, 11]	Felix had apples. Lauren gave him some apples. Felix now has apples. How many apples did Lauren give him? [3, 6] [9, 18] [14, 28]	Felix had some apples. Lauren gave him more apples. Now he has apples. How many apples did Felix start with? [5, 10] [7, 14] [12, 24]
Separate (Take From)	Felix had apples, but then he gave apples to Lauren. How many apples does Felix have now? [6, 3] [16, 8] [26, 13]	Felix had apples, and then he gave some to Lauren. Now Felix has apples. How many apples did he give to Lauren? [8, 4] [12, 6] [30, 15]	Felix had some apples. He gave to Lauren. Now he has apples left. How many apples did he have to start? [2, 2] [6, 6] [14, 14]
Part-Part-Whole (Put Together/ Take Apart)	Whole Unknown	Part Unknown	Both Parts Unknown
	Felix has red apples and green apples. How many apples does he have? [4, 4] [7, 7] [16, 16]	Felix has apples. apples are red and the rest are green. How many green apples does Felix have? [6, 3] [12, 6] [24, 12]	Felix has apples. He has the same number of red and green apples. How many does he have of each color? [8] [16] [28]
Compare	Difference Unknown	Compare Quantity Unknown	Referent Unknown
	Felix has apples. Lauren has apples. How many more apples does Felix have than Lauren? [4, 2] [18, 9] [22, 11]	Felix has apples. Lauren has more apples than Felix. How many apples does Lauren have? [3, 3] [6, 6] [13, 13]	Felix has apples. He has more apples than Lauren. How many apples does Lauren have? [8, 4] [18, 9] [30, 15]

Games & Purposeful Practice



Number Talk



Number Talk

$$2 + 2$$

$$3 + 3$$

$$4 + 4$$

$$5 + 5$$

Lesson String

3 Act Task

Image Talk

Anchor Problem

Tool Talk

Contextualized Practice Problems

Game

Number Talk

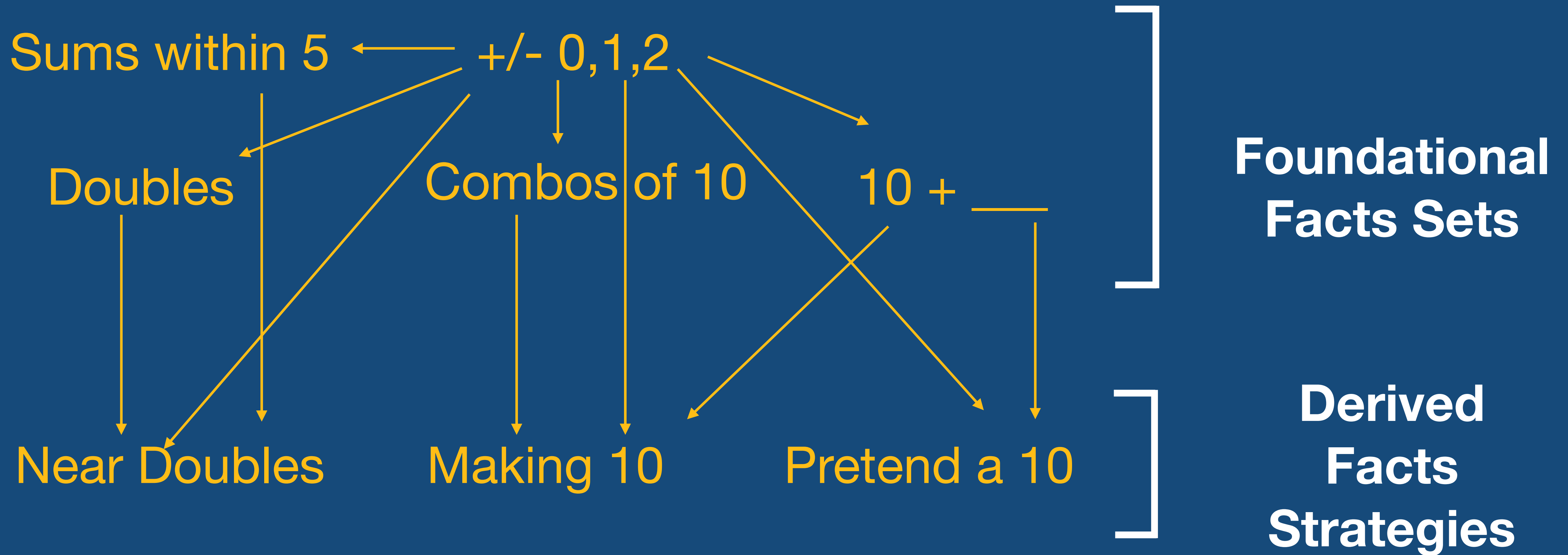


Fact fluency is an outcome of
meaningful problem-solving
with purposeful practice.



Addition/Subtraction Fact Fluency

Flexible Learning Progression



Properties of Addition

Commutative Property of Addition

Associative Property of Addition

Additive Identity Property of Zero



Commutative Property of Addition

+/- 0,1,2

$$2 + 9 = 9 + 2$$



Associative Property of Addition

Make a 10

$$8 + 7$$

$$8 + (2 + 5)$$

$$(8 + 2) + 5$$

$$10 + 5 = 15$$



Additive Identity Property of Zero

$$0 + a = a$$



Why?



Reasoning with facts beyond 20



Associative Property of Addition

$$8 + 7$$

$$8 + (2 + 5)$$

$$(8 + 2) + 5$$

$$10 + 5 = 15$$



Associative Property of Addition

$$68 + 7$$

$$68 + (2 + 5)$$

$$(68 + 2) + 5$$

$$70 + 5 = 75$$



Associative Property of Addition

$$5.8 + 7.4$$

$$5.8 + (0.2 + 7.2)$$

$$(5.8 + 0.2) + 7.2$$

$$6 + 7.2 = 13.2$$



Assessment



2 > 4



Formative
Diagnostic
Summative





What does Samora know?



Formative Assessment

Doubles

Nears Doubles

Use foundational facts to derive new facts

Connecting addition and multiplication

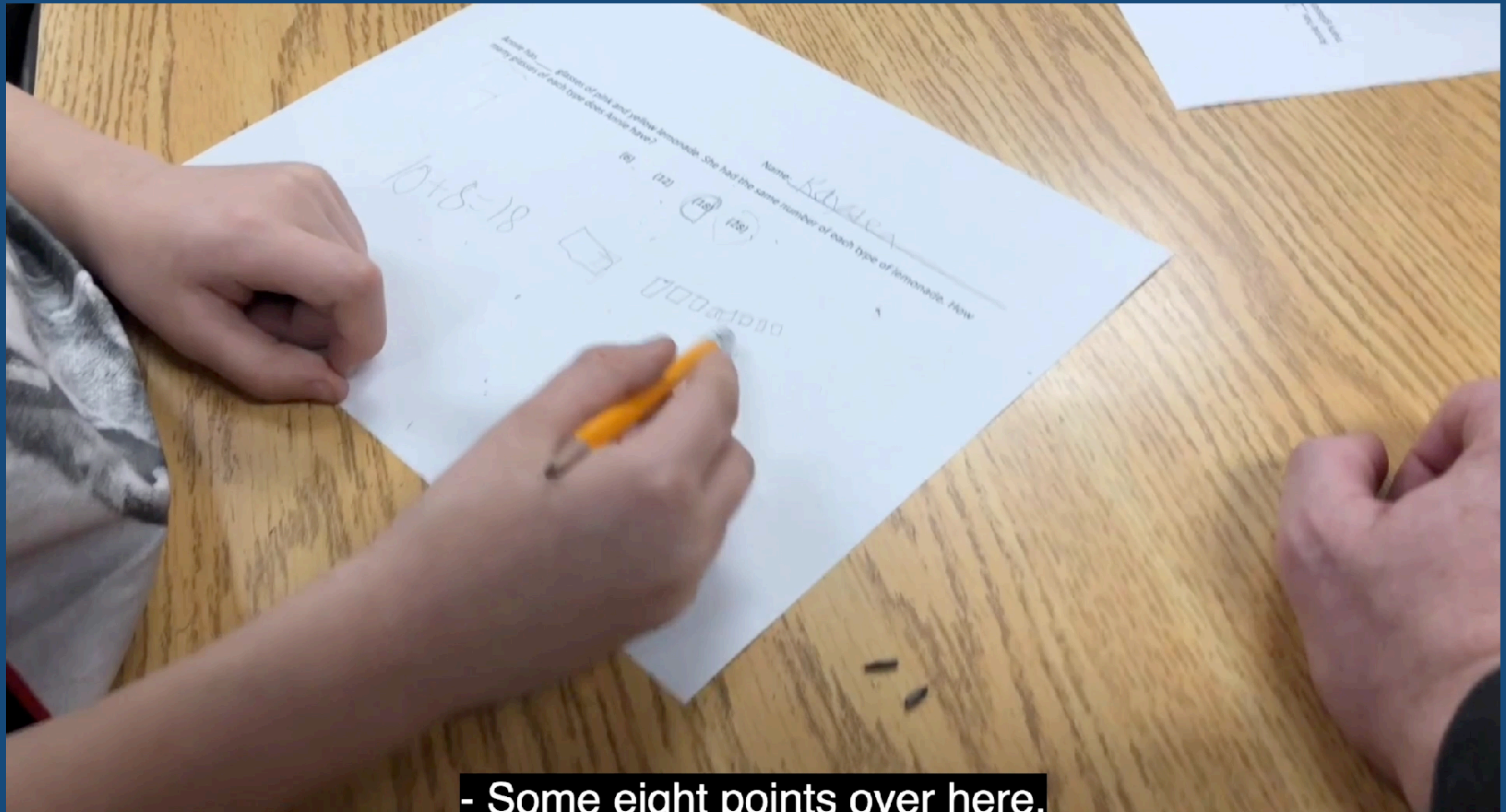
Annie has glasses of pink and yellow lemonade.
She has the same number of glasses of each type of
lemonade. How many glasses of each type does
Annie have?

[6]

[12]

[18]

[28]



- Some eight points over here.

Module 2

Part 1: Launch the Concept

Part 2: Connect and Extend Thinking

Part 3: Building Fluency



Building Fluency

Dominating Doubles & Near Doubles



Dominating Doubles (6-Sided Die)

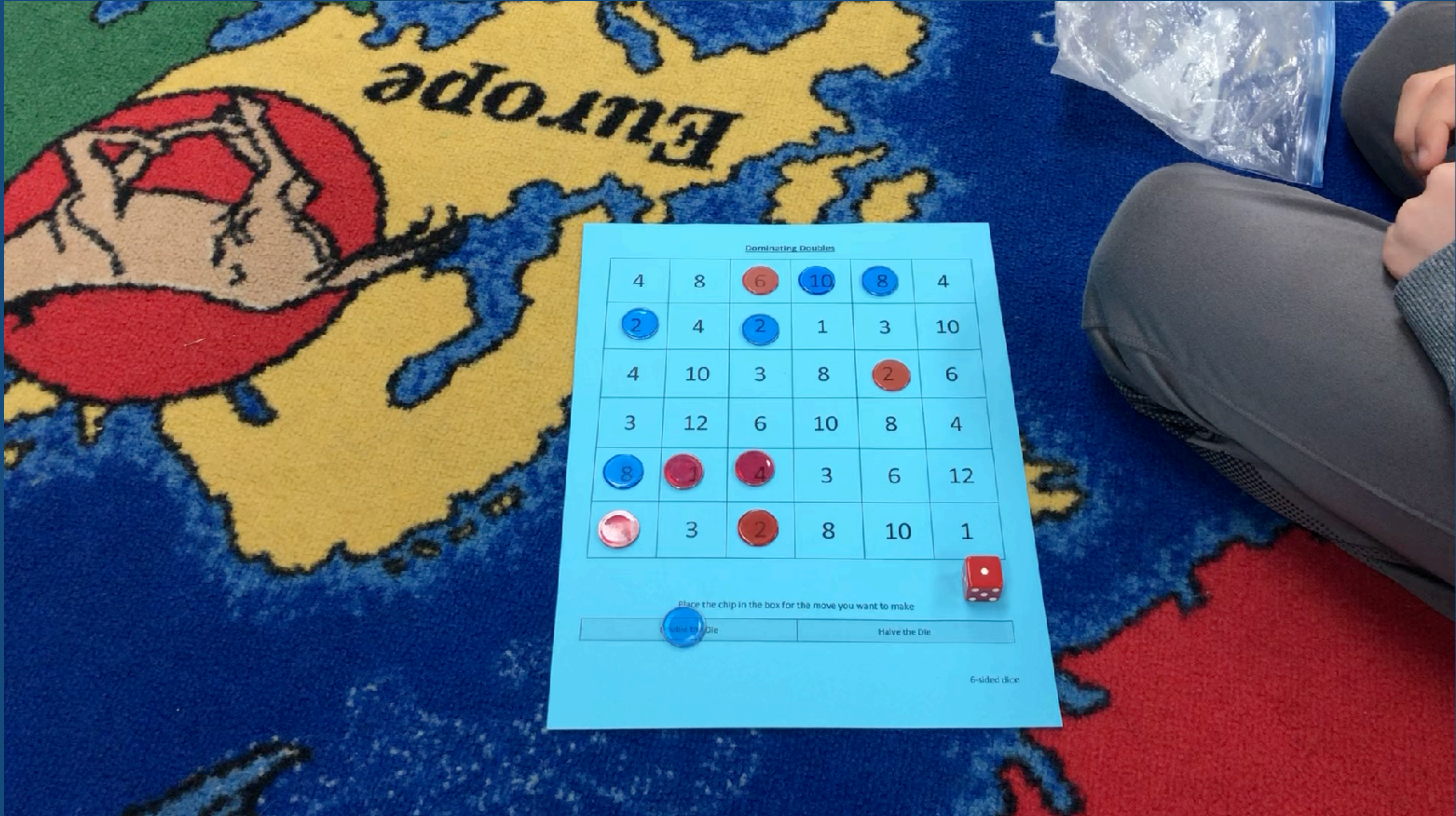


4	8	6	10	8	4
2	4	2	1	3	10
4	10	3	8	2	6
3	12	6	10	8	4
8	1	4	3	6	12
4	3	2	8	10	1

Place the chip in the box for the move you want to make.

Double the Die

Halve the Die



Dominating Doubles

4	8	6	10	8	4
2	4	2	1	3	10
4	10	3	8	2	6
3	12	6	10	8	4
8	1	4	3	6	12
1	3	2	8	10	1

Place the chip in the box for the move you want to make

Double the Die	Halve the Die
----------------	---------------

6-sided dice

Dominating Doubles (6-Sided Die)



4	8	6	10	8	4
2	4	2	1	3	10
4	10	3	8	2	6
3	12	6	10	8	4
8	1	4	3	6	12
4	3	2	8	10	1

Place the chip in the box for the move you want to make.

Double the Die	Halve the Die
----------------	---------------

Near Doubles within 19 (Doubles +/- 1)



7	8	9	10	11	12
13	4	12	16	17	18
19	6	13	14	15	19
1	2	9	8	11	10
8	17	7	13	4	5
10	11	3	17	12	6

Place the counter in the box for the move you want to make.

Double then subtract 1	Double	Double then add 1
------------------------	--------	-------------------

Building Fluency

Partners for Ten



Building Fluency

Make-10 or Pretend-10 with 8



Building Fluency

4-in a Row Addition & Subtraction



Building Fluency

Open Middle Problems



Building Fluency

Open Middle Cards

0	5	6	9
1	4	7	8
2	3	+	+
+	-	=	=
=	=	+	DO NOT USE DISCARD

Open Middle

Using the digits 1-9, complete the number sentences below.
Each number sentence should represent a near doubles fact.
Each digit can only be used one time.

$$\square + \square = \square$$

$$\square + \square = \square \square$$

Open Middle

Using the digits 1-9, complete the number sentences below.
Each number sentence should represent a near doubles fact.
Each digit can only be used one time.

$$\boxed{5} + \boxed{4} = \boxed{9}$$

$$\boxed{6} + \boxed{7} = \boxed{1} \boxed{3}$$

Open Middle

Using the digits 0-9 at most one time each fill in the boxes below to make a true statement.

$$\square + \square = \square + \square = \square + \square = \square + \square = \square - \square$$

Open Middle

Using the digits 0-9 at most one time each fill in the boxes below to make a true statement.

$$\boxed{1} + \boxed{8} = \boxed{2} + \boxed{7} = \boxed{3} + \boxed{6} = \boxed{5} + \boxed{4} = \boxed{9} - \boxed{0}$$

Building Fluency

Part Whole Cards



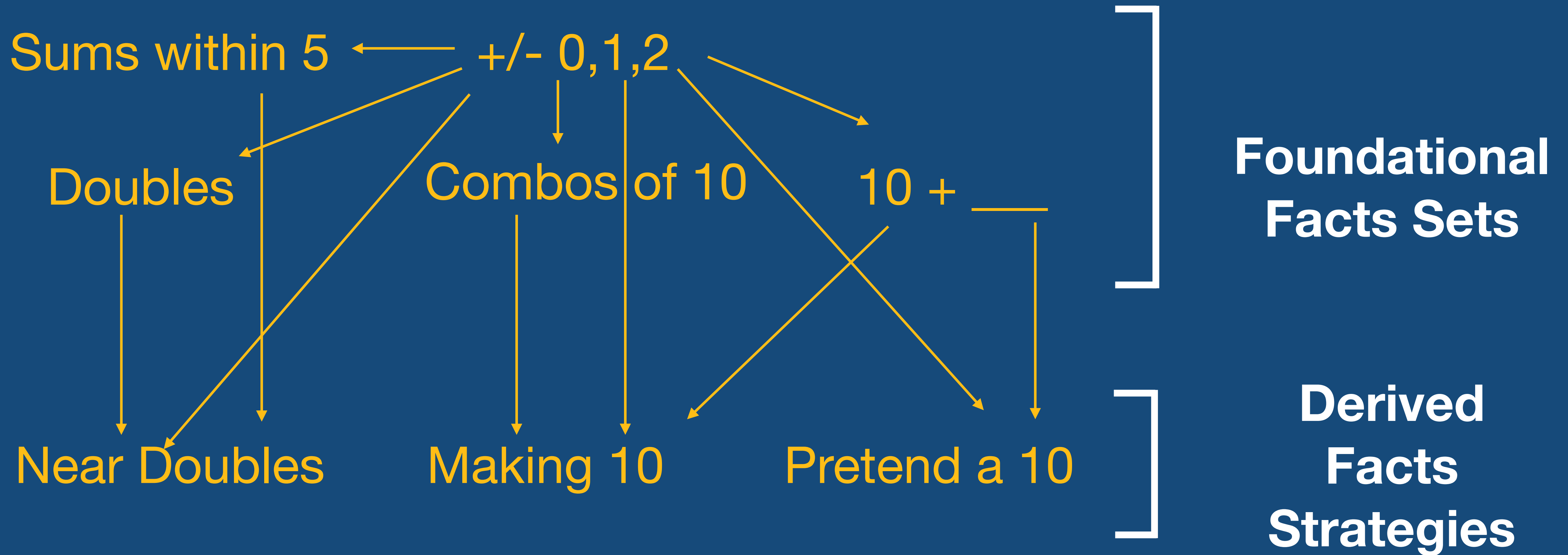
Building Fluency

Rekenrek & the Flexible Learning Progression



Addition/Subtraction Fact Fluency

Flexible Learning Progression



Coming Up.....

Week 1: Early Number and Counting

Week 2: Building Fluency through 20

Week 3: Building Fluency through 120

Week 4: Building Fluency through 1000

Week 5: Number and Measurement

Week 6: Number and Statistical Reasoning

Foundations of Numerical Reasoning in Grades K-2

Module 2: Building Fluency through 20

www.gfletchy.com



@gfletchy



gfletchy@gmail.com

