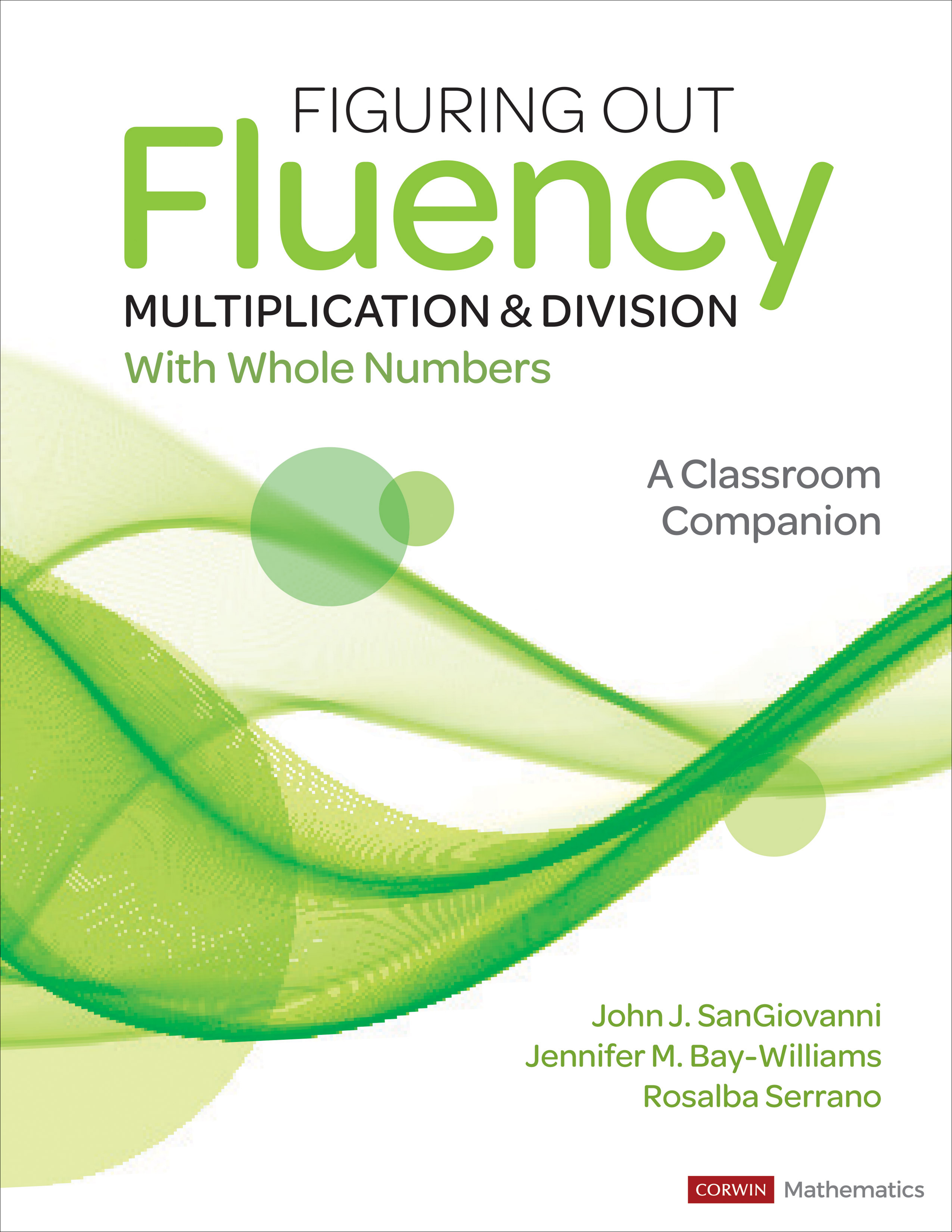


The background of the cover features abstract, flowing green lines that create a sense of movement. There are also several overlapping circles in different shades of green, some of which are semi-transparent, adding to the layered, organic feel of the design.

FIGURING OUT Fluency

MULTIPLICATION & DIVISION

With Whole Numbers

A Classroom
Companion

John J. SanGiovanni
Jennifer M. Bay-Williams
Rosalba Serrano

WHAT YOUR COLLEAGUES ARE SAYING . . .

“This text provides a clear definition of what fluency really is and provides strategies to deepen students’ number sense and make them fluent mathematicians.”

Meghan Schofield

Third-Grade Teacher

“I wish I’d had this book when I was in the classroom 10 years ago. The authors clearly lay out a pathway to procedural fluency, including intentional activities to understand and practice specific strategies, while also advocating for space for students to make decisions and feel empowered as mathematical thinkers and doers.”

Kristine M. Gettelman

Instructional Designer
CenterPoint Education

“This book is a must-read for teachers wanting to learn more about focused math fluency instruction. The steps are clear and easy to follow. You will have all the steps to help your students become fluent math thinkers.”

Carly Morales

Instructional Coach
District 93

“Are you ready to help your students connect their Number Talks and number routines to the real world? *Figuring Out Fluency* will give you the routines, games, protocols, and resources you need to help your students build their fluency in number sense (considering reasonableness, strategy selection, flexibility, and more). Our students deserve the opportunity to build a positive and confident mathematics identity. We can help support them to build this identity by providing them with access to a variety of strategies and the confidence to know when to use them.”

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Upper Grand District School Board

“*Figuring Out Fluency* goes beyond other resources currently on the market. It not only provides a robust collection of strategies and routines for developing fluency but also pays critical attention to the ways teachers can empower each and every student as a mathematical thinker who can make strategic decisions about their computation approaches. If you are looking for instruction and assessment approaches for fluency that move beyond getting the right answer, this is the resource for you.”

Nicole Rigelman

Professor of Mathematics Education
Portland State University

“As principal of a Title I school and former mathematics specialist, I appreciate this resource for what it is; a true teacher’s companion! The authors provide explicit strategy instruction in whole number multiplication and division to foster flexibility, efficiency, and true fluency in ALL students.”

Allie S. Watkins

Principal

Frederick County Public Schools

“This is the book classroom teachers have been waiting for! It provides extensive support as students gain confidence with multiplying and dividing whole numbers, and it does so with a deep esteem for educators and learners. The authors, highly respected in the field of mathematics education, provide careful explanations of effective strategies that will help students build their number sense and their computational fluency. There are routines for supporting instruction in the classroom, games and centers for practicing the strategies, and prompts to encourage sense making and extend learning. Kudos to all three authors for writing such an important, no-nonsense book. I cannot wait to share it with my colleagues!”

Elisa Waingort

Grades 4 & 5 Spanish Bilingual Teacher

Calgary Board of Education

“For years research has indicated that fluency is much more than speed, yet timed assessments and traditional instruction persist for teachers without a clear vision or tools to change their practices. This series provides teachers with the explicit examples, resources, and activities needed to bring that research to life for their students and will quickly become a well-worn guidebook for every fluency-focused classroom. This is the toolkit teachers have been yearning for in their journey toward fluency with their students.”

Gina Kilday

Elementary Math Interventionist and MTSS Coordinator

“Fluency isn’t a dry landscape of disconnected facts—it is a rich soil for developing and connecting diverse perspectives and ideas. This book series equips you with a deep understanding of fluency and a variety of activities to engage students in co-constructing ideas about multiplication and division that will last a lifetime.”

Berkeley Everett

Math Coach and Facilitator for UCLA Mathematics Project

Math Consultant for DragonBox

“This is the book we (educators) have been waiting for! *Figuring Out Fluency – Multiplication and Division With Whole Numbers* provides classroom teachers, special ed teachers, and instructional coaches with the support they need to develop true fluency for every child. This book is packed with great activities; quality practice that is not a worksheet; examples, routines, games, and centers; and a companion website with resources ready to use! Any question you may have—about how and when to teach a specific strategy, how to engage your students in meaningful practice, or assessing your students’ fluency—will be answered in this book! Any educator that is serious about understanding how basic fact

strategies grow into general reasoning strategies and how to advance their students' fluency will want this book! True fluency is a way of thinking, rather than a way of doing!"

Melisa Jean Hancock

Mathematics Consultant

"*Figuring Out Fluency – Multiplication and Division With Whole Numbers* is a must-have for all educators who teach multi-digit multiplication and division. With a focus on equity embedded throughout, this practical, ready-to-use resource provides everything needed to teach students to become strategic thinkers while giving *all* children access to reasoning."

Nichole DeMotte

Atkinson Academy

K–5 Mathematics Coach

"This book—indeed this *series*—is a must-read for elementary and middle level teachers, coaches, and administrators. Within this resource you will find a synthesis of important research organized to help readers develop a clear and common understanding of fluency paired with a large collection of teaching activities that provide concrete ways to support students' fluency development. *Figuring Out Fluency* provides a much-needed roadmap for teachers looking to increase computational proficiency with multiplication and division."

Delise Andrews

3–5 Mathematics Coordinator

Lincoln Public Schools

"The authors John J. SanGiovanni, Jennifer M. Bay-Williams, and Rosalba Serrano shine a bright light on how math fluency is *the* equity issue in mathematics education. How refreshing to have a book that equips math educators with the research and strategies to make a difference for *all* students! Let's implement these strategy modules in this book and help kids figure out fluency once and for all!"

Kelly DeLong

Executive Director for the Kentucky Center for Mathematics

Northern Kentucky University

"In *Figuring Out Fluency – Multiplication and Division With Whole Numbers*, John J. SanGiovanni, Jennifer M. Bay-Williams, and Rosalba Serrano have provided readers with a thorough education that guides them through the entire fluency journey. Grounded in research and packed with illustrative examples, this book is a must-have that delivers practical strategies, tools, resources, and recommendations that will immediately enhance your practice. In a sea of books on fluency, this one stands out. In a word . . . wow!"

Alison J. Mello

Assistant Superintendent Foxborough Public Schools

Author

Math Consultant

“John J. SanGiovanni, Jennifer M. Bay-Williams, and Rosalba Serrano hit the mark with *Figuring Out Fluency – Multiplication and Division With Whole Numbers*. The activities are fun and engaging and encourage students to think more flexibly and fluently as they work with multiplication and division strategies. Bravo!”

Joshua Barnes

Third-Grade Teacher

“The number one topic teachers grapple with every year is fluency. This companion book is an all-encompassing resource that takes the guess work out of multiplication and division fluency instruction and practice! With a focus on reasoning, understanding, and reasonableness, this is a practical, easy-to-follow guide for teachers. The strategy modules contain teaching prompts, routines, games, center activities, and a very critical component missing in most resources—strategy briefs for families.”

Marissa Walsh

Elementary Math Instructional Coach
Blue Springs School District

Figuring Out Fluency—Multiplication and Division With Whole Numbers: A Classroom Companion The Book at a Glance

Building off of *Figuring Out Fluency*, this classroom companion dives deep into five of the Seven Significant Strategies, plus the standard algorithms, that relate to procedural fluency when multiplying and dividing whole numbers, beyond basic facts.

FIGURE 13 Reasoning Strategies for Multiplying and Dividing Whole Numbers

REASONING STRATEGIES	RELEVANT OPERATIONS
1. Break Apart to Multiply (Module 1)	Multiplication
2. Halve and Double (Module 2)	Multiplication
3. Compensation (Module 3)	Multiplication
4. Partial Products and Quotients (Modules 4 and 6)	Multiplication and Division
5. Think Multiplication (Module 5)	Division

Strategy overviews and family briefs communicate how each strategy helps students develop flexibility, efficiency, accuracy, automaticity, and reasonableness.

MODULE 2

Halve and Double Strategy

STRATEGY OVERVIEW:
Halve and Double

What is Halve and Double? As the name implies, this strategy involves changing two factors, halving one of them and doubling the other. It is a useful strategy for multiplication facts. For example, 5s facts can be thought of as halving the other number and doubling 5:

$$5 \times 8 = 10 \times 4$$

The strategy also generalizes to larger numbers:

$$5 \times 18 = 10 \times 9$$

$$50 \times 48 = 100 \times 24$$

You can also halve and double more than once if continuing to use the strategy makes the problem easier to multiply:

$$25 \times 640 = 50 \times 320$$

$$50 \times 320 = 100 \times 160$$

$$= 16,000$$

A key to this strategy is being adept at halving numbers and doubling numbers. In other words, students need automaticity with halving and doubling.

HOW DOES HALVE AND DOUBLE WORK?

This strategy is a special case of the Break Apart Into Factors where one factor is broken apart into 2 × [the rest of the factor]. That 2 is associated with the factor, doubling it. It looks like this:

$$8 \times 35 = (8 \div 2) \times 35$$

$$= 4 \times (2 \times 35)$$

$$= 4 \times 70$$

$$= 280$$

WHEN DO YOU CHOOSE THE HALVE AND DOUBLE STRATEGY?

This strategy is very useful any time one factor is 5, 50, 500, and so on. When the other factor is even, it's all the more convenient, but the other factor does not have to be even for the strategy to work.

$28 \times 5 = 14 \times 10$
 $34 \times 50 = 17 \times 100$
 $79 \times 500 = 39.5 \times 1,000$

Doubling a number that ends in a 5, like 35, results in a tens number, like 70. This often makes the adapted problem easier to multiply, even if it continues to be a written problem and not a mental problem:

$$68 \times 350 = 34 \times 700$$

HALVE AND DOUBLE:
Strategy Brief for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. This strategy brief is a tool for doing that. You can include it in parent or school newsletters or share it at parent conferences. It is available for download so that you can adjust it as needed.

How It Works: We can halve a factor and double another to get an answer:

1. Choose which factor to halve and which to double.
2. Halve one factor.
3. Double the other factor.
4. Find the product by multiplying.

The left example shows that you can halve 14 and double 15. These new numbers make it easier to multiply. Now the number sentence is 7×30 , which is 210 . So, $14 \times 15 = 210$.

The right example shows a different way that you can halve and double. The problem is 25×8 . 8 can be halved and 25 doubled to create 50×4 . You can double and halve again: 4 can be halved and 50 doubled, resulting in 100×2 . $100 \times 2 = 200$ is the same as $25 \times 8 = 200$.

When It's Useful: Halving and doubling is useful when one factor is an even number. It also helps when both factors can be easily halved and doubled. For example, 28×12 is not a good fit for this strategy because 19×34 is not a friendly problem to solve.

HALVE

14×15

7×30

$7 \times 30 = 210$

HALVE

25×8

50×4

100×2

$50 \times 4 = 200$

$100 \times 2 = 200$

MODULE 1

Break Apart to Multiply Strategy

MODULE 2

Halve and Double Strategy

MODULE 3

Compensation Strategy

MODULE 4

Partial Products Strategy

MODULE 5

Think Multiplication Strategy

MODULE 6

Partial Quotients Strategy

MODULE 7

Standard Algorithms for Multiplication and Division

This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

44

Part 2 • Module 2 • Halve and Double Strategy 45

Each strategy module starts with teaching activities that help you explicitly teach the strategy.

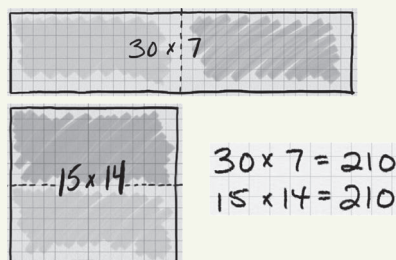
TEACHING ACTIVITIES for Halve and Double

Before students are able to choose strategies, a key to fluency, they first must be able to understand and use relevant strategies. These activities focus specifically on Halve and Double. Halve and Double is a strategy that works well in certain cases. While students may employ other methods, which is appropriate, they also must learn this strategy to develop their flexibility and efficiency. Students must understand that halving a factor while doubling the other yields the same result. You can develop this concept by working with a variety of representations and connecting them to equations. In this section, we focus on instructional activities for helping students develop understanding of the Halve and Double strategy and for determining when it is appropriate. Fluency with this strategy will build as students develop skill with halving or doubling numbers in general.



ACTIVITY 2.1 HALVING AND REARRANGING RECTANGLES

In this activity, you pose rectangles to students for them to trace on grid paper and cut out. Rectangle dimensions to use could include 15×14 , 28×5 , 16×6 , 25×22 , and 16×16 . Students record the equation for the area of one of their rectangles. Then, they cut the rectangle in half and rearrange it to make one, longer rectangle and record the equation for it. In the following figure, the original rectangle was 15×14 . The student cut it in half, creating two rectangles that were each 15×7 . Then, she aligned those two smaller rectangles, creating a rectangle that was 30×7 . She recorded the equations for each rectangle's area.



Students repeat halving and rearranging for the other rectangles that you pose. As students work, you want them to think about how the factors and the area of the corresponding rectangles are similar and different. After students complete the activity, a group discussion should be had about how one factor is halved and how the other is doubled yet the product (area) remains the same. Ask students to determine if this idea always works and then have them find examples to prove their case. It's possible that students will overlook that one factor is always even. If so, be sure to insert a rectangle, like 13×15 or 17×21 , for them to explore.

Even though this is an introduction to the concept of halving and doubling, it isn't too soon to discuss when this technique might be a good idea. Notice that 15×14 and 28×5 from the suggested set create expressions that yield multiples of 10 (30×7 and 14×10 , respectively). Because of this, the Halve and Double strategy is a good choice for these expressions.

Each strategy shares worked examples for you to work through with your students as they develop their procedural fluency.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

Halve and Double is useful in special cases, so a major challenge is to just remember it is a strategy and be on the lookout for when it will work. Other challenges include the following:

1. The student makes an error in halving an odd number.
 - 65×5 : changes to 38.5×10 .
2. The student has trouble choosing which number to halve and which to double.
 - 450×12 : changes to 225×24 , rather than 900×6 .

A key idea is to look for a number that, when doubled, is more manageable to multiply. In addition, noticing a second opportunity to use Halve and Double can continue to make the problem easier to work. Worked examples can attend to these ideas. The prompts from Activity 2.5 can be used for collecting worked examples. Throughout the module are various worked examples that you can use as fictional worked examples. A sampling of additional ideas is provided in the table.

SAMPLE WORKED EXAMPLES FOR HALVE AND DOUBLE

Correctly Worked Example (make sense of the strategy) What did _____ do? Why does it work? Is this a good method for this problem?	Trevor's work for 250×28 : 250×28 500×14 $1,000 \times 7 = 7,000$
Partially Worked Example (implement the strategy accurately) Why did _____ start the problem this way? What does _____ need to do to finish the problem?	Neesha's start for 15×260 : $30 \times 130 =$ Dante's start for 25×68 : $50 \times 34 =$
Incorrectly Worked Example (highlight common errors) What did _____ do? What mistake does _____ make? How can this mistake be fixed?	Greta's work for 50×45 : $100 \times 23.5 = 2,350$



ACTIVITY 2.6

Name: *"A String of Halves"*

Type: *Routine*

About the Routine: Making use of the Halve and Double strategy relies on students' ability to double and find halves. Often, students do well with doubling but finding a half proves more challenging. "A String of Halves" aims to help students improve their skill with halving. This routine uses a set of numbers to help students see how numbers are halved.

Materials: Prepare a set of numbers that are intentionally related.

- Directions:**
1. Post a set of three or more related numbers as shown.
 2. Ask students to mentally find the halves of as many numbers as possible.
 3. If students are unable to find all of the halves, have them talk with partners about how they can use the relationships between numbers and the known halves to find the unknown half. If students find all of the halves, have them discuss with partners how the numbers and their halves are related. To extend this situation (when all halves are known), have students generate a new number and its half that is related to the set.

A	40	16	56
	20	8	28

These two examples of "A String of Halves" were used in a fourth-grade classroom. The teacher started with Example A, recording 40, 16, and 56 because she wanted to develop an idea about how to halve 56. Students were asked to think of the half for each of the numbers. They could halve 40 and 16. At that point, she asked students to talk with partners about how they could figure out the half of 56 by knowing the halves of 40 and 16. The teacher then used Example B in a similar way. She posed the three related numbers, and students found the halves. She then asked students to look for patterns and relationships between the numbers and the halves to prove what half of 30 is.

Halving numbers like 76 or even 90 can be challenging for students because there is an element of regrouping within the half. This routine can be leveraged for developing skill with those halves by making use of relationships and patterns. In the following figure, four multiples of 10 (60, 70, 80, and 90) are posed. Students are tasked with halving each, but the conversation focuses first on 60 and 80 before students work to find half of 70 and 90. During these discussions, focus their attention on half of 10 being added to or taken from half of 60 or 80 and why. Of course, another strategy is to think of the half (35) between those halves (30 and 40).

C	60	70	80	90
	30		40	

Routines, Games, and Centers for each strategy offer extensive opportunity for student practice.

Download the resources you need for each activity at this book's companion website.



ACTIVITY 2.10

Name: *Halve and Double Flips*

Type: *Game*

About the Game: *Halve and Double Flips* practices identifying multiplication problems that can be solved with the Halve and Double strategy. Players place chips on corresponding spaces and can flip opponents' pieces that lie directly between two of their pieces (similar to *Othello*).

Materials: *Halve and Double Flips* game cards and game board, multiplication expression cards, and two-colored counters for game pieces

- Directions:**
1. Players take turns pulling a multiplication expression card.
 2. Players consider if the expression can be solved more efficiently using the Halve and Double strategy.
 3. If the expression can be solved more efficiently with the strategy, the players put their counter on any space labeled "WORKS" (i.e., the Halve and Double strategy works for this expression).
 4. If the expression cannot be solved more efficiently with the strategy, the player puts their counter on any space labeled "DOESN'T WORK" (i.e., the Halve and Double strategy doesn't make the problem easier to solve).
 5. When placing a game piece, any opponent pieces between a previously placed piece and the current piece are flipped (similar to *Othello*). Note that unlike *Othello*, a player does not have to place a piece at the end of a row of pieces.
 6. The game ends when all of the spaces are filled.
 7. The player with the most chips on the board wins the game.

RESOURCE(S) FOR THIS ACTIVITY

44×5	14×15
3×16	50×18
40×28	32×6
16×6	18×40
28×5	5×48



Game cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

FIGURING OUT
Fluency
MULTIPLICATION & DIVISION
With Whole Numbers

Grades
K–8

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A Classroom Companion

John J. SanGiovanni
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resources.corwin.com/FOF/multiplydividewholenumber for
 downloadable resources.

Preface

Fluency is an equity issue. In written documents and in our daily work, we (mathematics teachers and leaders) communicate that every student must be *fluent* with whole number multiplication and division, for example. But we haven't even come close to accomplishing this for each and every student. The most recent National Assessment of Educational Progress (NAEP) data, for example, finds that about two-fifths (41%) of the nation's Grade 4 students are at or above proficient and about one-third (34%) of our nation's Grade 8 students at or above proficient (NCES, 2019). We can and must do better! One major reason we haven't been able to develop fluent students is that there are misunderstandings about what fluency really means.

FIGURING OUT FLUENCY

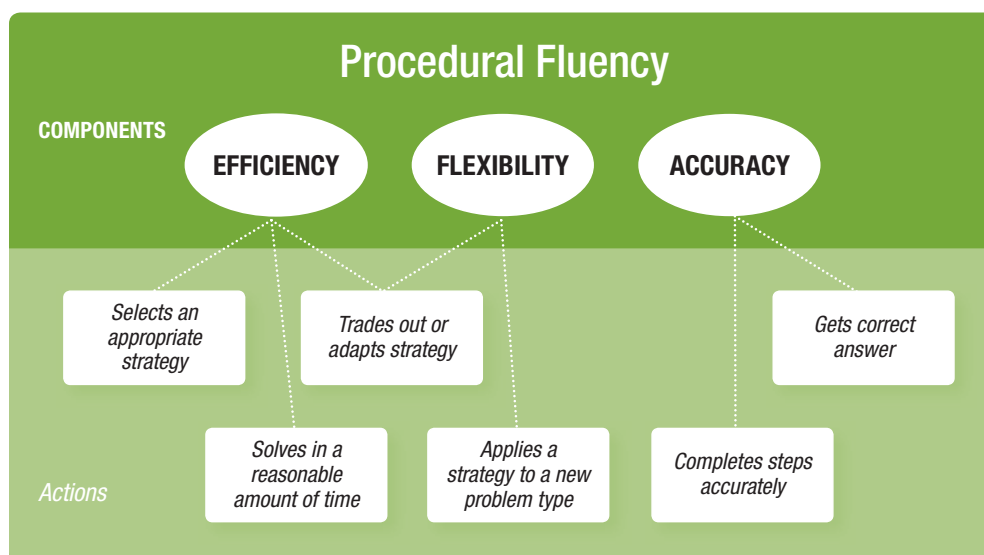
In order to ensure every student develops fluency, we first must:

- Understand what procedural fluency is (and what it isn't),
- Respect fluency, and
- Plan to explicitly teach and assess reasoning strategies.

If you have read our anchor book *Figuring Out Fluency for Mathematics Teaching and Learning*—which we recommend in order to get the most out of this classroom companion—you'll remember an in-depth discussion of these topics.

WHAT PROCEDURAL FLUENCY IS AND ISN'T

Like fluency with language, wherein you decide how you want to communicate an idea, fluency in mathematics involves decision-making as you decide how to solve a problem. In our anchor book, we propose the following visual as a way to illustrate the full meaning of fluency.



Source: Adapted with permission from D. Spangler & J. Wanko (Eds.), *Enhancing Classroom Practice with Research behind Principles to Actions*, copyright 2017, by the National Council of Teachers of Mathematics. All rights reserved.

Part 1 of this book explains the elements of this procedural fluency graphic. To cut to the chase, procedural fluency is much more than knowing facts and standard algorithms. Fluency involves higher-level thinking, wherein a person analyzes a problem, considers options for how to solve it, selects an efficient strategy, and accurately enacts that strategy (trading it out for another if it doesn't go well). Decision-making is key and that means you need to have good options to choose from. This book provides instructional and practice activities so that students learn different options (Part 2) and then provides practice activities to help students learn to choose options (Part 3).

RESPECT FLUENCY

We are strong advocates for conceptual understanding. We all must be. But there is not a choice here. Fluency relies on conceptual understanding, and conceptual understanding alone cannot help students fluently navigate computational situations. They go together and must be connected. Instructional activities throughout Part 2 provide opportunities for students to discuss, critique, and justify their thinking, connecting their conceptual understanding to their procedural knowledge and vice versa.

EXPLICITLY TEACH AND ASSESS REASONING STRATEGIES

If every student is to be fluent in whole number multiplication and division, then every student needs access to the significant strategies for these operations. And there must be opportunities for students to learn how to select the best strategy for a particular problem. For example, students may learn that the Compensation strategy works well when one of the factors is close to, but less than, a ten or hundred (e.g., 49×8 or 398×7). To accomplish this, all three fluency components must have equitable attention in instruction and assessment. This is a major shift from traditional teaching and assessing, which privileges accuracy over the other two components and the standard algorithm over reasoning strategies.

Let's unpack the phrase *explicit strategy instruction*. According to the *Merriam-Webster Dictionary*, explicit means “fully revealed or expressed without vagueness” (“Explicit,” 2021). In mathematics teaching, being explicit means making mathematical relationships visible. A strategy is a flexible method to solve a problem. Explicit strategy instruction, then, is engaging students in ways to clearly see how and why a strategy works. For example, with multiplication, students might compare 12×15 to 6×30 with equal groups (12 groups of 15 as compared to 6 groups of 30) and use rectangles (cutting the rectangle in half and moving the half to double one side) to see how the Halve and Double strategy works. Showing that $12 \times 15 = 6 \times 30$ and proving the generalization of this Halve and Double strategy engages students in higher-level thinking while helping them understand the strategy. For division, students might be asked to explore the conjecture: You can break apart a dividend into a sum, divide it into parts, put it back together and get the same answer. (e.g., For $84 \div 7$, break apart 84 into $70 + 14$. Divide these parts by 7, equaling $10 + 2$, and put them back together to equal 12.) Once understood, students need to explore *when* a

strategy is a good option. Learning how to use and how to choose strategies empowers students to be able to decide how they want to solve a problem, developing a positive mathematics identity and a sense of agency.

USING THIS BOOK

This book is a classroom companion to *Figuring Out Fluency for Mathematics Teaching and Learning*. In that anchor book, we lay out what fluency is, identify the fallacies that stand in the way of a true focus on fluency, and elaborate on necessary foundations for fluency. We also propose the following:

- Seven Significant Strategies across the operations, five of which apply to whole number multiplication and division
- Eight “automaticities” beyond automaticity with basic facts, five of which are relevant to whole number multiplication and division
- Five ways to engage students in meaningful practice
- Four assessment options that can replace (or at least complement) tests and that focus on real fluency
- Many ways to engage families in supporting their child’s fluency

In Part 1 of this book, we revisit some of these ideas in order to connect specifically to whole number multiplication and division. This section is not a substitute for the anchor book, but rather a brief revisiting of central ideas that serve as reminders of what was fully illustrated, explained, and justified in *Figuring Out Fluency in Mathematics Teaching and Learning*. Hopefully, you have had the chance to read and engage with that content with colleagues first, and then Part 1 will help you think about those ideas as they apply to whole number multiplication and division. Finally, Part 1 includes suggestions for how to use the strategy modules.

Part 2 is focused on explicit instruction of each significant strategy for whole number multiplication and division. Each module includes the following:

- An overview for your reference and to share with students and colleagues
- A strategy brief for families
- A series of instructional activities, with the final one offering a series of questions to promote discourse about the strategy
- A series of practice activities, including worked examples, routines, games, and center activities that engage students in meaningful and ongoing practice to develop proficiency with that strategy

As you are teaching and find your students are ready to learn a particular strategy, pull this book off the shelf, go to the related module, and access the activities and ready-to-use resources. While the modules are sequenced in a developmental order overall, the order and focus on each strategy may vary depending on your grade and your students’ experiences. Additionally, teaching within or across modules does not happen all at once; rather, activities can be woven into your instruction regularly, over time.

Part 3 is about becoming truly fluent—developing flexibility and efficiency with multiplication and division of whole numbers. Filled with more routines, games, and centers, the focus here is on students *choosing* to use the strategies that make sense to them in a given situation. Part 3 also provides assessment tools to monitor students’ fluency. As you are teaching and find your students are needing opportunities to choose from among the strategies they are learning, pull this book off the shelf and select an activity from Part 3.

In the Appendix, you will find lists of all the activities in order to help you easily locate what you are looking for by strategy or by type of resource.

This book can be used to complement or supplement any published mathematics program or district-created program. As we noted earlier, elementary mathematics has tended to fall short in its attention to efficiency and flexibility (and the related Fluency Actions illustrated in the earlier graphic). This book provides a large collection of activities to address these neglected components of fluency. Note that this book is part of a series that explores other operations and other numbers. You may also be interested in *Figuring Out Fluency for Whole Number Addition and Subtraction* as well as the classroom companion books for decimal and fraction operations.

WHO IS THIS BOOK FOR?

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With 120 activities and a companion website with resources ready to download, this book is designed to support classroom teachers as they advance their students’ fluency with whole number multiplication and division. Special education teachers will find the explicit strategy instruction, as well as the additional practice, useful in supporting their students. Mathematics coaches and specialists can use this book for professional learning and to provide instructional resources to the classroom teachers they support. Mathematics supervisors and curriculum leads can use this book to help them assess fluency aspects of their mathematics curriculum and fill potential gaps in resources and understanding. Teacher preparation programs can use this book to galvanize preservice teachers’ understanding of fluency and provide teacher candidates with a wealth of classroom-ready resources to use during internships and as they begin their career.

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PART 1

FIGURING OUT FLUENCY

Key Ideas

WHAT IS FLUENCY WITH WHOLE NUMBER MULTIPLICATION AND DIVISION?

To set the stage for figuring out fluency for whole number multiplication and division, take a moment to do some math. Find the solution to each of these.

$$35 \times 19$$

$$2,458 \div 8$$

$$756 \div 7$$

$$320 \times 5$$

$$240 \div 3$$

$$24 \times 15$$

How did you find the products and quotients? Did you use the same approach or strategy for each? Did you move between different strategies? Did you change out a strategy based on the numbers within the problem? Did you start with one strategy and shift to another? You likely said “yes” to all of these questions because you fluently multiply and divide whole numbers. Yet another reader can answer “yes” to each of these questions as well but solve each problem differently. This is true because fluency is a way of thinking rather than a way of doing. Thinking is unique to each individual. Thinking is grounded in parameters but its execution is left to the understanding, preference, and creativity of each thinker.

Of course, there are strategies used most frequently for certain problems (based on the numbers in the problem), but even in those examples, efficient alternatives are likely. For example, 320×5 may seem like a problem that “fits” Partial Products (Module 4). Find the product of 300×5 (1,500) and 20×5 (100) and add them together (1,600). But a Halve and Double approach (Module 2) may “jump out” to another person who sees 320×5 as 160×10 . Which method is closer to your way of thinking? Or did you think about it differently?

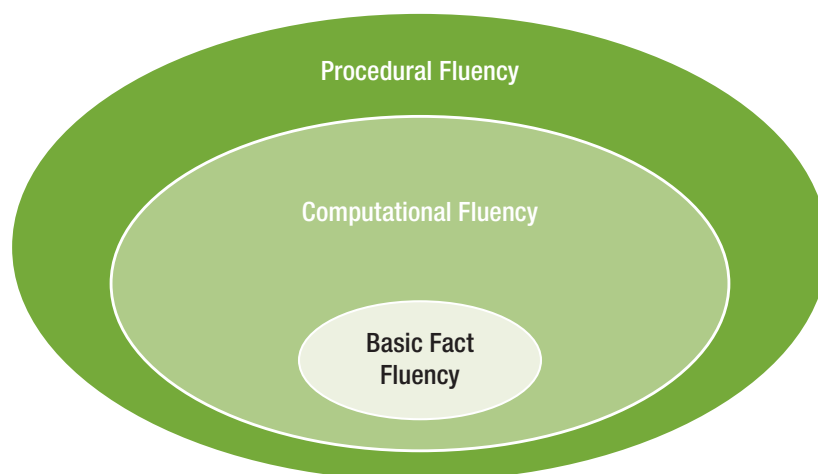
Real fluency is the ability to select efficient strategies; to adapt, modify, or change out strategies; and to find solutions with accuracy. Real fluency is not the act of replicating someone else’s steps or procedures for doing mathematics. It is the act of thinking, reasoning, and doing mathematics on one’s own. Before fluency can be taught well, you must understand what fluency is and why it matters.

TEACHING TAKEAWAY

Real fluency is the ability to select efficient strategies; to adapt, modify, or change out strategies; and to find solutions with accuracy.

Procedural fluency is an umbrella term that includes basic fact fluency and computational fluency (see Figure 1). *Basic fact fluency* attends to fluently adding,

FIGURE 1 • The Relationship of Different Fluency Terms in Mathematics



subtracting, multiplying, and dividing single-digit numbers. *Computational fluency* refers to the fluency in four operations across number types (whole numbers, fractions, etc.), regardless of the magnitude of the number. Procedural fluency encompasses both basic fact fluency and computational fluency, plus other procedures like finding equivalent fractions.

Beyond being an umbrella term that encompasses basic fact and computational fluency, procedural fluency is well defined as solving procedures efficiently, flexibly, and accurately (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2014). These three **components** are defined as follows:

Efficiency: solving a procedure in a reasonable amount of time by selecting an appropriate strategy and readily implementing that strategy

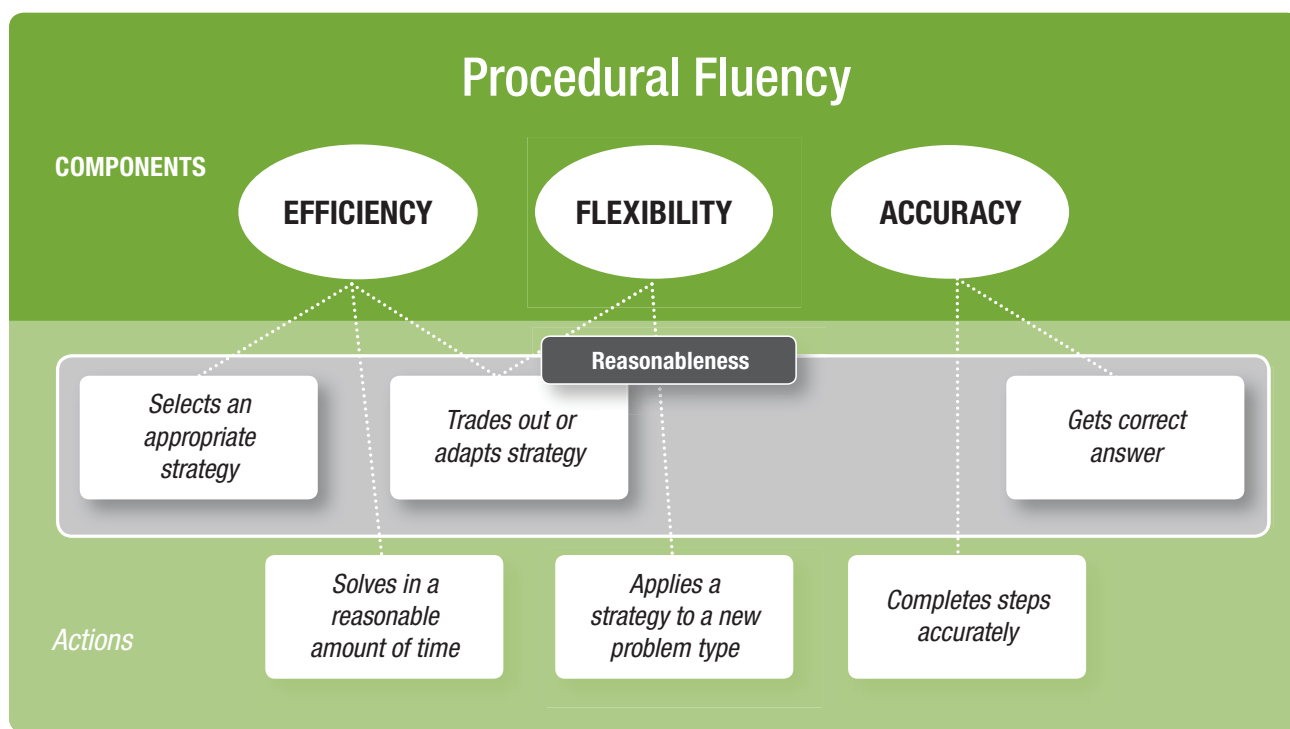
Flexibility: knowing multiple procedures and can apply or adapt strategies to solve procedural problems (Baroody & Dowker, 2003; Star, 2005)

Accuracy: correctly solving a procedure

Strategies are not the same as algorithms. Strategies are general methods that are flexible in design, compared to algorithms that are established steps implemented the same way across problems.

To focus on fluency, we need specific observable actions that we can look for in what students are doing in order to ensure they are developing fluency. We have identified six such actions. The three components and six Fluency Actions, and their relationships, are illustrated in Figure 2.

FIGURE 2 ● Procedural Fluency Components, Actions, and Checks for Reasonableness



Source: Adapted with permission from D. Spangler & J. Wanko (Eds.), *Enhancing Classroom Practice with Research behind Principles to Actions*, copyright 2017, by the National Council of Teachers of Mathematics. All rights reserved.

Three of the six Fluency Actions (should) attend to reasonableness. Fluency Actions and reasonableness are described later in Part 1, but first, it is important to consider why this “bigger” (comprehensive) view of fluency matters.

WHY FOCUS ON FLUENCY FOR WHOLE NUMBER MULTIPLICATION AND DIVISION?

There are two key reasons why it’s important to focus on fluency with whole number multiplication and division. First, it is a critical foundation for ensuring that students fully realize procedural fluency in general, with all kinds of numbers. Fluency with whole numbers begins with developing fluency with single-digit multiplication and division (the basic facts) and seeing how strategies such as Subtract-a-Group can be transferred to using Compensation in order to multiply 29×8 efficiently (e.g., 29×8 is adjusted to 30×8 and to compensate, 8 is subtracted from the product). Basic facts are addressed in more detail later in Part 1.

Second and most importantly, developing fluency is an equity issue. Equipping students with options for how to solve multiplication and division problems and positioning students to choose a method that works best for them develops a positive mathematics identity and sense of agency. Conversely, trying to remember algorithms and feeling anxiety about being correct or fast develops a negative mathematics identity and lack of agency.

WHAT DO FLUENCY ACTIONS LOOK LIKE FOR WHOLE NUMBER MULTIPLICATION AND DIVISION?

The six Fluency Actions are observable and therefore form a foundation for assessing student progress toward fluency. Let’s take a look at what each of these actions looks like in the context of whole number multiplication and division.

FLUENCY ACTION 1: Select an Appropriate Strategy

TEACHING TAKEAWAY

Selecting *an* appropriate strategy does not mean selecting *the* appropriate strategy. Many problems can be solved efficiently in more than one way.

Selecting *an* appropriate strategy does not mean selecting *the* appropriate strategy. Many problems can be solved efficiently in more than one way. Here is our operational definition: *Of the available strategies, the one the student opts to use gets to a solution in about as many steps and/or about as much time as other appropriate options.*

Consider 25×38 . One could leverage the distributive property by breaking apart factors and adding the partials as shown in Figure 3a, in which 25 is decomposed into $10 + 10 + 5$ and the products of those numbers and 38 are added together. One could break apart 38 by factors using 2×19 and rethink 25×38 as $25 \times 2 \times 19$, which becomes 50×19 (as shown in Figure 3b). Figure 3c shows Compensation in which one thinks of a friendlier problem, $25 \times 40 = 1,000$, and then compensates for the two extra groups of 25 by taking them from 1,000. Notice that this Fluency Action is one of three connected to *reasonableness*. Within this action is noticing when a strategy “fits” the numbers in the problem. This particular problem fits well with Compensation because 38

is close to (but less than) 40. The other strategies are reasonable if you are adept at multiplying by 5s. In other words, what is an “appropriate” strategy depends on the problem and on the person (their knowledge and experiences).

FIGURE 3 • Multiplying 25×38

a. Break Apart to Multiply
(break apart into addends)

$$\begin{array}{r} 25 \times 38 \\ 10 \times 38 = 380 \\ 10 \times 38 = 380 \\ 5 \times 38 = 190 \\ \hline 950 \end{array}$$

b. Break Apart to Multiply
(break apart into factors)

$$\begin{array}{r} 25 \times 38 \\ 25 \times 2 \times 19 \\ 50 \times 19 = 950 \end{array}$$

c. Compensation
(change a factor and adjust the answer)

$$\begin{array}{r} 25 \times 38 \\ 25 \times 40 = 1,000 \\ 25 \times 2 = 50 \\ 1,000 - 50 = 950 \end{array}$$

A strategy cannot be used until it is understood. Once understood, a strategy becomes part of a student’s repertoire of options and they are then able to select the strategy.

Importantly, we name strategies so that we can talk about them. But one approach may fit within various types of strategies and have different names. For example, solving 23×9 as $20 \times 9 + 3 \times 9$ may be using Break Apart thinking or Partial Products thinking. The focus must be on the ideas (not the naming of the strategy).

FLUENCY ACTION 2: Solve in a Reasonable Amount of Time

There is no set amount of time that should be expected for solving any whole number multiplication or division problem. Students should be able to work through a problem without getting stuck or lost. The amount of time is relative to the student’s grade and mathematical maturity. Keep in mind that appropriate strategies can be carried out in inefficient, unreasonable ways. For example, Figure 3a shows a Break Apart approach that can be enacted in a reasonable amount of time. In contrast, Figure 4 shows this approach with many more parts, which requires significantly more time. The approach is likely the result of the student’s ability to recall or find certain products ($\times 10$, $\times 2$). While this may be an appropriate beginning strategy, it is ultimately not an approach that can be completed in a reasonable amount of time.

FIGURE 4 • Solving 25×38 by Breaking Apart 38

$$\begin{array}{r} 25 \times 38 \\ 10 \times 25 = 250 \quad 2 \times 25 = 50 \\ 10 \times 25 = 250 \quad 2 \times 25 = 50 \\ 10 \times 25 = 250 \quad 2 \times 25 = 50 \\ \hline 750 \quad 2 \times 25 = 50 \\ \hline 200 \\ 25 \times 38 = 950 \end{array}$$

FLUENCY ACTION 3: Trade Out or Adapt a Strategy

As students' number sense and understanding of strategies advances, they are able to adapt and trade out strategies. Let's revisit 25×38 . A student might attempt 25×38 as illustrated in Figure 3a but get stuck because they don't know 5×38 . Thus, they decide to *adapt* the strategy by breaking apart 38 instead:

$$\begin{aligned} 25 \times 30 &= 750 \\ 25 \times 8 &= 200 \\ 750 + 200 &= 950 \end{aligned}$$

Or a student might *trade out* the Break Apart strategy for another strategy. For example, the student might use Compensation (see Figure 3c) or Partial Products (two options are illustrated in Figure 5a). Notice that this Fluency Action is one of three connected to *reasonableness*. Within this action is noticing how the use of the strategy is going. If it isn't going well or if a student is getting bogged down, then the strategy needs to be adapted or traded for another, more efficient option.

FIGURE 5 • Partial Products

a. Partial Products Decomposing Both Factors b. Partial Products Decomposing One Factor

25×38 $\begin{array}{r} 20 \times 30 = 600 \\ 5 \times 30 = 150 \\ 20 \times 8 = 160 \\ 5 \times 8 = 40 \\ \hline 950 \end{array}$	25×38 $\begin{array}{r} 30 \times 25 = 750 \\ 8 \times 25 = 200 \\ \hline 950 \end{array}$
---	---

FLUENCY ACTION 4: Apply a Strategy to a New Problem Type

The Compensation work shows how, when multiplying 25×38 , 38 can be thought of as 40 to find $25 \times 40 = 1,000$ and then two groups of 25 can be taken away. That individual's first experience with compensation was likely finding products of $\times 9$ basic facts that transferred to multidigit factors that were one group away from a friendly computation such as 18×9 or 35×39 . In time, this strategy will be applied to multiplying fractions by thinking of $6 \times 2\frac{3}{4}$ as 6×3 , taking away $6 \times \frac{1}{4}$ ($18 - 1\frac{1}{2} = 16\frac{1}{2}$) or multiplying decimals by thinking of 6×1.9 as 6×2 and taking away six tenths ($12 - 0.6 = 11.4$).

FLUENCY ACTIONS 5 AND 6: Complete Steps Accurately and Get Correct Answers

These two Fluency Actions are about accuracy. An error at the end of a problem may be due to an error in how a strategy was enacted. For example, in

using Compensation for 25×38 , a student changes the problem to 25×40 to get 1,000 and then *subtracts* 2 from the answer, resulting in the answer 998. This incorrect answer is due to a misconception of the steps in implementing the Compensation strategy (specifically not recognizing that they need to compensate by subtracting two groups of 12). Conversely, a student may enact the strategy accurately but make a computational error. The student's work in Figure 6 is a good example of this. Here, the student correctly enacts the steps for Partial Products but makes an error with 20×8 that leads to the wrong answer.

FIGURE 6 ● Incorrect Partial Product Example

$$\begin{array}{r}
 25 \times 38 \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 20 \quad 5 \quad 30 \quad 8 \\
 \\
 20 \times 30 = 600 \\
 20 \times 8 = 1,600 \\
 5 \times 30 = 150 \\
 5 \times 8 = 40 \\
 \hline
 9,500
 \end{array}$$

Fluency Action 6 is one of three connected to *reasonableness*. Within this action is noticing if your answer makes sense. While reasonableness has been woven into the discussion of Fluency Actions, it is critical to fluency and warrants more discussion.

REASONABLENESS

As described earlier, reasonableness is more than “checking your answer.” Reasonableness occurs in three of the six Fluency Actions as shown in Figure 2 and described within the related Fluency Actions. Let’s revisit 25×38 . This example has several reasonable options, including Break Apart to Multiply, Compensation, and Halve and Double (Action 1). Skip-counting by 25s or 38s is not a reasonable strategy. Let’s say a student chooses Break Apart (using addends). They break apart 25 ($20 \times 38 + 5 \times 38$) but don’t know these partial products. The student recognizes that this attempt is not going well and adapts or trades out the strategy (Action 3). Finally, they notice that 950 is a reasonable answer (Action 6). Knowing that $25 \times 38 = 950$ is reasonable can be determined in (at least) three different ways:

1. You might estimate the product to be less than 1,200 because both factors round up to 30 and 40, respectively.
2. You might estimate it to be about 1,000 using friendly numbers (25×40).
3. You might think about it being in the range of 600 (20×30) and 1,200 (30×40).

It takes time to develop reasonableness. It should be practiced and discussed as often as possible. Students can develop reasonableness by practicing three moves (a match to the Fluency Actions 1, 3, and 6).

THREE 'Cs' OF REASONABLENESS

Choose:

Choose a strategy that is efficient based on the numbers in the problem.

Change:

Change the strategy if it is proving to be overly complex or unsuccessful.

Check:

Check to make sure the result makes sense.

You can encourage and support student thinking about reasonableness by providing Choose, Change, Check reflection cards (see Figure 7). These cards can be adapted into anchor charts for students to use while working on problems or during class discussions about multiplying and dividing.

FIGURE 7 ● Choose, Change, Check Reflection Card for Students

CHECKS FOR REASONABLENESS		
Choose 	Change 	Check 
Is this something I can do in my head? What strategy makes sense for these numbers?	Is my strategy going well or should I try a different approach? Does my answer so far seem reasonable?	Is my answer close to what I anticipated it might be? How might I check my answer?

Icon sources: Choose by iStock.com/Enis Aksoy; Change by iStock.com/Sigit Mulyo Utomo; Check by iStock.com/Indigo Diamond



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

WHAT FOUNDATIONS DO STUDENTS NEED TO DEVELOP FLUENCY WITH MULTIPLICATION AND DIVISION?

To develop fluency, students need a strong foundation in five domains:

- *Conceptual understanding*: knowing the meaning of the operations
- *Properties*: being able to use the operations in order to manipulate numbers and retain equivalence
- *Utilities*: small skills that make a big difference, such as knowing or easily finding half of a number
- *Computational estimation*: being able to quickly and flexibly determine a “close” answer
- *Basic facts*: single-digit addition, subtraction, multiplication, and division facts that are needed for multidigit work

Rushing students to strategy instruction before these foundations are firmly in place can be disastrous.

CONCEPTUAL UNDERSTANDING

Developing fluency from conceptual understanding begins with including concrete experiences for students so they can make sense of the quantities. Hence, developing fluency *begins* with stories. It is a mistake to save story problems as an application, as stories give students a context from which they can reason.

As students work with multiplication and division story problems, there are few important ideas to keep in mind:

1. Stories need to be *relevant* to students, meaning that students are familiar with the context and it is interesting to students.
2. Avoid key-word strategies for solving story problems so that students must make sense of the problem and how multiplication or division can be used to find a solution.
3. Balance experiences with division between both partitive and quotative (measurement) interpretations of division.
4. Vary and balance experience with the different situations for multiplication and division as shown in Figure 8.
5. Use representations that are true to the situation (e.g., equal group representations for equal group problems, area representations for area problems) until students have a deep understanding and are able to work symbolically and move between representations and tell explicitly how they are related.

Figure 8 can be used as a resource to be sure you are varying your story types (for example, you can tally which types of stories you are telling as part of action research).

TEACHING TAKEAWAY

Developing fluency *begins* with stories and contexts. It is a mistake to save story problems as an application, as stories give students a context from which they can reason.

FIGURE 8 • Multiplication and Division Situations

MULTIPLICATION AND DIVISION SITUATIONS			
Equal Groups	Area/Array	Compare	Combinations
Story is about a quantity of same-sized grouped amount.	Story is about a quantity of equal-length rows.	Story compares two quantities multiplicatively.	Story is about finding how many pairings are possible (and beyond).
Ex: AJ has ____ stacks of books. Each stack has ____ books. She has a total of ____ books.	AJ has ____ books on each shelf. She has ____ shelves. She has a total of ____ books.	AJ has ____ books. Ian has ____ books. AJ has ____ times more books than Ian.	AJ has ____ books and ____ magazines. She takes one of each to school. There are ____ different combinations.

In addition to stories, visuals provide concrete, conceptual beginnings for students. For example, young children count collections of objects in order to start learning to skip count (Franke, Kazemi, & Turrou, 2018). Counting objects progresses to counting visuals and representations, which eventually leads to abstract counting strategies and representation of groups.

PROPERTIES AND UTILITIES FOR STRATEGIC COMPETENCE

TEACHING TAKEAWAY

Knowing properties does not equal using properties.

In addition to conceptual foundations, fluency is grounded in using properties of the operations and a few other skills that we refer to as “utilities” because students must utilize them in their reasoning. First, fluency with multiplication relies heavily on students using the commutative, associative, and distributive properties of multiplication. Note that knowing properties does not equal using properties. It is not useful to have students simply name the associative property. It is absolutely necessary that students utilize this property in solving problems efficiently. For example, in the following problem, students break apart factors (44) by their factors (4×11) and rearrange these numbers (mentally or in writing) using the commutative property to create friendly computations.

$$\begin{array}{l}
 44 \times 25 \\
 4 \times 11 \times 25 \\
 4 \times 25 \times 11 \\
 100 \times 11 \\
 1,100
 \end{array}$$

The distributive property is most critical for fluency with multiplication because so many strategies, including the standard algorithm, make use of it. On the left in the following example, the distributive property is applied to 8×398 in order to find partial products. Numerically, this is $8 \times 398 = 8(300 + 90 + 8)$. On the right, the distributive property is used for a compensation strategy. Numerically, this is $8 \times 398 = 8(400 + -2) = 8(400 - 2)$.

$$\begin{array}{r}
 8 \times 398 \\
 \begin{array}{|c|c|c|}
 \hline
 300 & 90 & 8 \\
 \hline
 8 & 2,400 & 720 & 64 \\
 \hline
 \end{array} \\
 3,184
 \end{array}$$

$$\begin{array}{l}
 8 \times 398 \\
 8 \times 400 = 3,200 \\
 - 8 \times 2 = 16 \\
 8 \times 398 = 3,184
 \end{array}$$

See Chapter 3 (pp. 47–75) of *Figuring Out Fluency* for more about foundations and good beginnings for fluency.

Beyond the properties is a short list of utilities that support fluency, which is presented in Figure 9.

FIGURE 9 • Utilities for Strategic Competence With Multiplication and Division

UTILITY	WHAT IT IS	RELATIONSHIP TO FLUENCY
Distance From a 10	Knowing that 9 is 1 away from 10 (and 8 is 2 away and so on).	Knowing how far a number (e.g., 5, 6, 7, 8, 9) is from 10 is necessary for finding how many groups of a number are away from a friendly computation (e.g., 28 is 2 away from 30, so 28×6 is two groups of 6 away from 30 groups of 6).
Composing and Decomposing Numbers Flexibly	Understanding diverse, flexible ways to compose and decompose, including but not limited to place value decomposition.	Flexibly decomposing numbers supports strategy selection and facility with any of the strategies.
Skip Counting	Skip counting by multiples of tens, hundreds, and thousands as well as other useful benchmarks such as 25.	Efficiency comes from skip-counting with multiples. This applies to all multiplication and division strategies.
Multiplying and Dividing by Tens, Hundreds, and Thousands	Understanding that 6×40 is similar to 6×4 , in that 6×40 is saying six groups of 4 tens, which is 24 tens or 240.	Flexible decomposition of factors relies on recognition and knowing of products of factors that are multiples of 10, 100, or 1,000.

COMPUTATIONAL ESTIMATION

Just like computation, there are strategies for estimation and the use of those strategies should be *flexible*. For multiplication and division of whole numbers, students might use any of these methods:

1. *Rounding*: Flexible rounding means that one or both numbers might be rounded. Students may round to the nearest number or they may round one number up and one number down to have a more accurate estimate. Rounding is a well-known strategy but is often approached in a step-by-step manner, which can interfere with the point of estimating—getting a quick idea of what the answer will be close to. Use conceptual language, such as, “Which tens/hundreds/thousands is that number close to?” Help students understand that they choose how to round. For example, for 24×34 , rounding both to the nearest 10 will give a low estimate, whereas rounding one up and one down gives a closer estimate.
2. *Front-end estimation*: In its most basic form, students just multiply or divide using the largest place value. More flexibly, though, students may use the largest two place values or adjust their estimate because of what they notice with the rest of the numbers. Front-end estimation is *quick*. For example, 74×55 is about 3,500 (relying on 70×50). To adjust, take a quick look to the right and decide to keep estimate or adjust. With division, that front end must focus on compatibles. For example, to estimate $4,576 \div 7$, the front end is altered to be 4,200 or 4,900 because these “fronts” are multiples of 7 (this is an overlap to the next strategy, compatible numbers!).
3. *Compatible numbers*: With flexibility in mind, students change one or both of the numbers to a nearby number so that the numbers are easy to multiply or divide. For example, estimating the quotient of $337 \div 72$ could be to think of 337 as 350 and 72 as 70 to estimate a quotient of about 5. Compatibles are particularly useful with division. Consider estimating $345 \div 8$. Compatible alternatives might be $320 \div 8$, $400 \div 8$, or $350 \div 7$.

DEVELOPING BASIC FACT FLUENCY

The teaching of basic facts must attend to conceptual understanding and strategies for reasoning rather than rote instruction (Bay-Williams & Kling, 2019; O’Connell & SanGiovanni, 2014). Reasons to focus on *strategies* when teaching the basic facts (as opposed to memorizing) include the following:

1. It is well established across many studies that students actually learn and retain their facts better when they focus on conceptual understanding versus memorization. In fact, students don’t just learn and retain their facts better, they perform better in math in *general* (e.g., Baroody, Purpura, Eiland, Reid, & Paliwal, 2016; Brendefur, Strother, Thiede, & Appleton, 2015; Jordan, Kaplan, Ramineni, & Locuniak, 2009; Locuniak & Jordan, 2008; Purpura, Baroody, Eiland, & Reid, 2016).
2. Students need to know and use these strategies to support whole number multiplication and division (as well as decimal and fraction operations). In our *Figuring Out Fluency* anchor book, we elaborate more on the key strategies and ideas for effectively developing basic fact fluency.
3. Students who learn to use and choose strategies for basic facts develop confidence. Students who memorize often develop anxiety. A student who knows how to generate an answer to a quotient such as $9 \div 6$ (beyond counting) doesn’t have to worry if they forget the fact. This sense of agency is critical to student success in mathematics!

Figure 10 lists the basic fact strategies for multiplication and division. To be clear, automaticity is the goal for learning basic facts. Students become automatic through learning the strategies and practicing them over and over again. In so doing, students develop automaticity with the facts *and with implementing the strategies*. Examples are illustrated in Figure 11. Keep in mind that there are many ways to implement a reasoning strategy, and only one way is shown for each example.

FIGURE 10 • Reasoning Strategies for Basic Fact Multiplication and Division

STRATEGY NAME	HOW THE STRATEGY WORKS	EXAMPLE STUDENT TALK
Multiplication	Example: 6×7	
Doubling	Student sees an even factor, finds the product of half of that factor, and doubles their answer.	I got 42. I know 3 times 7 is 21 and I doubled 21.
Add-a-Group	Student thinks of a known fact where one of the factors is one less, multiplies, then adds a group back on.	When I see a 6, I use my 5s: 5 times 7 is 35 and 7 more is 42.
Subtract-a-Group	Student thinks of a known fact where one of the factors is one more, multiplies, then adds a group back on.	I know 7 groups of 7 is 49, so I subtract one group of 7 and I have 42.
Near Squares	Student uses a square fact they know and then adds or subtracts a group. <i>Note: This is an undertaught but useful strategy.</i>	Well, 6 times 6 is 36, and I add 6 more and get 42.
Division	Example: $36 \div 9$	
Think Multiplication	Student thinks, <i>How many groups of 9 make 36?</i>	I know 9 times 4 is 36, so it’s 4. Or I used doubling to get to 18, doubled again, and got 36, so it is 4.

Basic fact strategies evolve directly to the significant reasoning strategies for multidigit multiplication and division, which are the focus of Part 2 of this book. You can see this transformation in the examples in Figure 11.

FIGURE 11 • How Basic Fact Strategies Grow Into General Reasoning Strategies

REASONING STRATEGY	EXAMPLES WITH MULTIDIGIT NUMBERS
Add-a-Group(s) leads to Break Apart to Multiply , and Partial Products strategies	$42 \times 6 \rightarrow (40 \times 6) + (2 \times 6)$
Subtract-a-Group(s) leads to Compensation	$79 \times 5 \rightarrow (80 \times 5) - (1 \times 5)$
Doubling applies to the Halve and Double strategy and extends to Break Apart to Multiply , involving breaking apart into factors	$36 \times 5 \rightarrow 18 \times 10$ $45 \times 6 \rightarrow 45 \times 2 \times 3 \rightarrow 90 \times 3$ $25 \times 68 \rightarrow 25 \times 4 \times 17 \rightarrow 100 \times 17$
Think Multiplication extends to larger numbers and to Computational Estimation involving division.	$49 \div 7 \rightarrow 490 \div 7$ [or $490 \div 70$] $4871 \div 85 \rightarrow$ About how many 8s are in 48? [6, so estimate is 60]

WHAT AUTOMATICITIES DO STUDENTS NEED BEYOND THEIR BASIC FACTS?

Unlike the foundation of conceptual understanding, automaticities are not prerequisites for, but coincide with, strategy instruction. For example, automaticity with basic facts (just discussed) begins with strategy instruction and leads to eventual automaticity with the facts. But there are automaticities beyond the basic facts that support student reasoning!

Automaticity is the ability to complete a task with little or no attention to process. Little thought, if any, is given to skills that are automatic (Cheind & Schneider, 2012). We consider automaticities to be those skills that a fluent person can do without much attention to process. For example, you know that four 25s are 100 and a drawing or repeated addition is not needed; it is intuitive or reflexive. Figure 12 identifies automaticities that are particularly critical for multiplying and dividing whole numbers. Of course, this is not a complete list. These automaticities are strengthened through strategy instruction (and conversely, having these automaticities strengthens students' capacities to use strategies).

See Chapter 5 (pp. 107–129) of *Figuring Out Fluency* for more about automaticities for fluency.

FIGURE 12 ● Automaticities for Multiplying and Dividing Whole Numbers

AUTOMATICITY	WHAT IT IS	HOW IT COMPLEMENTS STRATEGY INSTRUCTION
Basic facts	Quickly recognizing how a problem relates to a basic fact (e.g., 30×80 relates to 3×8).	Identifying relationships to basic facts helps students consider which numbers to decompose and how to decompose them.
Using 25s	Knowing multiples of 25 (within reason) and how they are related to multiples of 250, 2,500, and so on.	This helps students think about how to decompose factors and find partials when multiplying and dividing.
Using 15s and 30s	Knowing multiples of 15 and 30 and ways to decompose them efficiently (e.g., 75 is 60 and 15 or two 30s and 15 or five 15s).	This helps students think about how to decompose factors and find partials when multiplying and dividing.
Doubling	Doubling a given number.	Although explicitly connected to the Halve and Double strategy, Doubling is useful for Compensation and strategies with partials.
Halving	Finding a half of a given number.	Though explicitly connected to the Halve and Double strategy, Halving is useful for Compensation and strategies with partials.

WHAT ARE THE SIGNIFICANT STRATEGIES FOR MULTIPLYING AND DIVIDING WHOLE NUMBERS?

Teaching strategies beyond the common algorithms has been a challenge, as there has been pushback and criticism from families and in social media. Two questions require attention:

1. Why do students need strategies when they can use the standard algorithm?

One way to quickly respond to this question is to share an example for which the standard algorithm takes much more time than an alternative. Examples include 24×5 , $125 \div 5$, or a problem that has various possibilities, like 16×35 . Why learn other methods? Because many problems can be solved more efficiently another way. Fluent students look for efficient methods; if students are limited in the methods they are taught, they have little to choose from, which limits flexibility and efficiency.

TEACHING TAKEAWAY

Students don't need to constantly learn dozens of new strategies, but rather connect how the key strategies they learned for basic facts are transferred to other numbers.

2. What strategies are worthy of attention?

Let's just take some pressure off here. The list is short, and we must help students see that they are not necessarily learning a *new* strategy, but they are applying a strategy they learned with basic facts and transferring it to other numbers. In *Figuring Out Fluency* we propose Seven Significant Strategies. Of these, five relate to multiplying and/or dividing whole numbers and they are listed in Figure 13.

FIGURE 13 ● Reasoning Strategies for Multiplying and Dividing Whole Numbers

REASONING STRATEGIES	RELEVANT OPERATIONS
1. Break Apart to Multiply (Module 1)	Multiplication
2. Halve and Double (Module 2)	Multiplication
3. Compensation (Module 3)	Multiplication
4. Partial Products and Quotients (Modules 4 and 6)	Multiplication and Division
5. Think Multiplication (Module 5)	Division

Using an area model or a number line is *not* a strategy. It is a representation. If a student multiplies using an area model, they are implementing a strategy—perhaps Break Apart or Partial Products. When a student says, “I used an area model,” ask *how* they used it—then you will learn what strategy they used. Teaching for fluency means that each of these strategies is explicitly taught to students. We teach students to *use* the strategy, and then we give students many opportunities to engage in *choosing* strategies (Part 3 of this book). Explicitly teaching a strategy does not mean turning the strategy into an algorithm. Strategies require flexible thinking. Each module provides instructional ideas and practice to ensure students become adept at using each strategy flexibly. There is also a module on **standard algorithms** (Module 7), so that they are integrated into the use of strategies.

HOW DO I USE THE PART 2 MODULES TO TEACH, PRACTICE, AND ASSESS STRATEGIES?

Part 2 is a set of modules, each one focused on understanding why a specific reasoning strategy works and learning how to use it well. The seven modules in Part 2 each have a consistent format. First, each module provides an overview of the strategy—unpacking what it is, how it works, and when it is useful. Then, each module provides a series of instructional activities for explicit strategy instruction, followed by a collection of practice activities, including routines, games, and centers.

EXPLICIT STRATEGY INSTRUCTION

Strategies must be explicitly taught so that students understand them and can use them. Each module provides core teaching activities for explicit instruction. The activities are designed so that you can modify and extend them as needed. Any one activity might form the focus of your instruction over the course of multiple lessons. Keep in mind that you can swap out tools and representations as well as adjust the numbers within the task.

The last teaching activity in each module is a collection of *investigation prompts* that you can use to develop reasoning and understanding of the strategy. Each investigation prompt itself can easily become a core teaching task.

We intend for students to work with instructional activities in collaborative partner or group settings. We encourage you to let students make their own

meaning and to make mistakes. After students engage in the activities, a group discussion is needed to focus student thinking on the concepts within the strategy, how the strategy works, and the different ways a strategy might be carried out.

QUALITY PRACTICE

Students need access to quality practice that is not a worksheet. Quality practice is focused on a strategy, varied in type of engagement, processed by the student to make sense of what they did, and connected to what they are learning.

See Chapter 6 (pp. 130–153) of *Figuring Out Fluency* for more about quality practice.

Each practice section begins with **worked examples**. Worked examples are opportunities for students to attend to the thinking involved with a strategy, without solving the problem themselves. We feature three types to get at all components of fluency:

1. *Correctly worked example*: efficiency (selects an appropriate strategy) and flexibility (applies strategy to a new problem type)
2. *Partially worked example*: efficiency (selects an appropriate strategy) and accuracy (completes steps accurately; gets correct answer)
3. *Incorrectly worked example*: accuracy (completes steps accurately; gets correct answer)

Also, comparing two correctly worked examples is very effective in helping students learn to choose efficient methods. Throughout the modules are dozens of examples, which can be used as worked examples (and adapted to other similar worked examples). Your worked examples can be from a fictional “student” or authentic student work. Some of the prompts from teaching the strategy section are, in fact, worked examples.

The remaining practice activities include routines, games, and centers. Each activity provides a brief “About the Activity” statement to help you quickly match what your students need with a meaningful activity. Game boards, recording sheets, problem cards, and all other activity resources, including modified versions, are available as online resources that you can download, modify, print, and copy. General resources, including number cards, mini ten-frame cards, multiplication charts, and more, are also available for download on the companion website.

ASSESSING STRATEGY USE

Each module offers a plethora of practice activities. As students are practicing, you can observe and assess the extent to which they are able to apply the selected strategy. **Observation tools** help you keep track of where each student is and monitor their progress. An observation tool can be simple, such as a class list with an extra column. Your observations can be codes:

- + Is regularly implementing the strategy adeptly
- ✓ Understands the strategy, takes time to think it through
- Is not implementing the strategy accurately

A note-taking observation tool provides space for you to insert notes about how a student is doing (see Figure 14). You can laminate the tool and use dry-erase markers to reuse it for different observations, use sticky notes, or just write in the boxes.

FIGURE 14 • Example Note-Taking Observation Tool

Strategy: <u>Break Apart to Multiply</u>		Type of Numbers: <u>two-digit × two-digit</u>	
R. Attis		R. Barnes	
Using rectangles and breaking apart in 4 parts		Excellent at breaking apart one of the factors	
			G. Carmes
			Needs an illustration
H. Moore		M. Franke	
Doing a lot mentally and quickly		Has difficulty choosing how to break apart	
			M. Gillis
			Is using flexible ways to break apart, not just place
C. Kruz		W. Munez	
Solving mentally with ease			
			P. Smith
			Tends to break apart first number
M. Roberts		M. Rodriguez	
		Solving quickly using some written steps	
			S. Sanchez
			Always changes both numbers



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

Some days, you collect data on some students; other days, you collect data on other students. The data can help in classroom discussions and in planning for instructional next steps.

Journal prompts provide an opportunity for students to write about their thinking process. Each module provides a collection of prompts that you might use for journaling. You can modify those or easily craft your own. The prompt can specifically ask students to explain how they used the strategy:

Explain and show how you can use Compensation to solve 6×498 .

Or a prompt can focus on identifying when that strategy is a good idea:

Circle the problems that are good choices for solving with the Compensation strategy and tell why you selected them:

$$495 \times 3$$

$$7 \times 726$$

$$440 \times 2$$

$$399 \times 6$$

Interviewing is an excellent way to really understand student thinking. You can pick any problem that lends to the strategy you are working on and write it on a note card (or record two or three on separate notecards). While students are engaged in an instructional or practice activity, roam the room, select a child, show them a card, and ask them (1) to solve it and (2) explain how they thought about it. You can pair this with an observation tool to keep track of how each student is progressing.

HOW DO I USE PART 3 TO SUPPORT STUDENTS' FLUENCY?

As soon as students know more than one way, it is time to integrate routines, tasks, centers, and games that focus on choosing when to use a strategy. That is where Part 3 comes in. As you read in Fluency Action 1, students need to be able to choose efficient strategies. The strategy modules provide students access to those strategies, ensuring the strategies make sense and giving students ample opportunities to practice those strategies and become adept at using them. However, if you stop there, students are left on their own when it comes to choosing which strategy to use when. It is like having a set of knives in the kitchen but not knowing which ones to use for slicing cheese or bread, cutting meat, or chopping vegetables. Like with food, some items can be cut with various knives, but other food really needs a specific knife.

Do not wait until after all strategies are learned to focus on when to use a strategy—instead weave in Part 3 activities regularly. Each time a new strategy is learned, it is time to revisit activities that engage students in making choices from among the strategies in their repertoire. Students must learn what to look for in a problem to decide which strategy they will use to solve the problem *efficiently* based on the numbers in the problem. This is *flexibility* in action, and thus leads to fluency.

TEACHING TAKEAWAY

Teaching for fluency means teaching strategies as core instruction, routinely practicing them, and offering opportunities for students to choose among strategies.

“FACTORS” IN GETTING THE BEST “PRODUCT”

Part 1 has briefly described factors that are important in developing fluency, and these ideas are important as you implement activities from the modules. We close Part 1 with five key factors to figuring out fluency.

1. Be clear on what fluency means (three components and six actions). This includes communicating it to students and their parents.
2. Attend to readiness skills: conceptual understanding, properties, utilities, computational estimation, and, of course, basic fact fluency.
3. Through activities and discussion, help students connect on the features of a problem and how that relates to good strategy options.
4. Reinforce student reasoning and choice selection, rather than focus on speed and accuracy. Getting the strategies down initially takes more time, but eventually will become more efficient.
5. Assess fluency, not just accuracy.

Time invested in strategy work has big payoffs—confident and fluent students (and that is the “best product”). That is why we have so many activities in this book. Teach the strategies as part of core instruction, *and* continue to practice throughout the year, looping back to strategies that students might be forgetting to use (with Part 2 activities) and offering ongoing opportunities to choose from among strategies (with Part 3 activities).

[illegible]

PART 2

STRATEGY MODULES

Break Apart to Multiply Strategy

STRATEGY OVERVIEW:

Break Apart to Multiply

What is Break Apart to Multiply? As the name implies, this strategy breaks apart (decomposes) one or both of the factors. They may be decomposed into addends (or “by addends”) or broken apart into factors (or “by multiplication”). The strategy begins with multiplication facts, as illustrated here:

STRATEGY	STUDENT THINKING FOR 6×7	SYMBOLIC REPRESENTATION
Add-a-Group (Break Apart Into Addends)	“I don’t know this fact, but I know 5×7 equals 35. I can break apart 6 into $5 + 1$. I know 5×7 is 35, and one more group of 7 equals 42.”	6×7 $(5 + 1) \times 7$ $5 \times 7 + 1 \times 7$ $35 + 7$ 42
Doubling (Break Apart Into Factors)	“I don’t know this fact, but I know 3×7 equals 21; 6 is double 3, so I can double 21. The answer is 42.”	6×7 $(2 \times 3) \times 7$ $2 \times (3 \times 7)$ 2×21 42

Let’s look at these two ways to Break Apart for the two-digit by two-digit problem of 25×16 :

BREAK APART A FACTOR INTO ADDENDS	BREAK APART A FACTOR INTO FACTORS
$ \begin{array}{c} 25 \times 16 \\ \swarrow \quad \searrow \\ 25 \times (10 + 6) \\ 250 + 150 \end{array} $	$ \begin{array}{c} 25 \times 16 \\ \swarrow \quad \searrow \\ 25 \times 4 \times 4 \\ 100 \times 4 \end{array} $

Notice that Break Apart Into Addends is an overlap to Partial Products and may be referred to either way. Halve and Double is an example of Break Apart Into Factors.

HOW DOES BREAK APART TO MULTIPLY WORK?

Breaking apart numbers into addends utilizes the distributive property. Breaking apart a factor into smaller factors and reassociating a factor utilizes the associative property, and sometimes the commutative property. The earlier examples show how one factor can be broken apart, but you can break apart both factors. By addends, this is $(20 + 5)(10 + 6)$ and an area model can be used to show the four partial products (this is what is actually happening in the standard algorithm). For breaking apart by factors, 25×16 can be decomposed into $5 \times 5 \times 4 \times 4$ as follows:

$$\begin{array}{cc} 25 & \times & 16 \\ / & & / \\ 5 & \times & 5 & & 4 & \times & 4 \end{array}$$

$$5 \times 4 \times 5 \times 4$$

$$20 \times 20$$

WHEN DO YOU CHOOSE BREAK APART TO MULTIPLY?

This strategy is a good choice when you are able to break apart a factor and then compute the parts mentally. For two-digit by one-digit multiplication, Break Apart by Addends is almost always a good option. Break Apart Into Factors is useful when reassociating a factor such as a 2, 4, or 5 leads to a benchmark. In the earlier example, the student noticed the **25** and knew (1) that $25 \times 4 = 100$ and (2) that 16 has 4 as a factor. As problems get larger, Break Apart is not always an efficient strategy. The key is to look at each factor and consider which one can be decomposed into addends or factors and how they can be decomposed so that they can be multiplied efficiently if not mentally.

MODULE

1

Break Apart to Multiply Strategy

MODULE

2

Halve and Double Strategy

MODULE

3

Compensation Strategy

MODULE

4

Partial Products Strategy

MODULE

5

Think Multiplication Strategy

MODULE

6

Partial Quotients Strategy

MODULE

7

Standard Algorithms for Multiplication and Division

BREAK APART TO MULTIPLY: Strategy Briefs for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. These strategy briefs are a tool for doing that. You can include them in parent or school newsletters or share them at parent conferences. They are available for download so that you can adjust them as needed.

Break Apart by Factors to Multiply

How It Works: We can break apart either factor into its factors and find the product of the original problem.

1. Choose which factor to break apart by its factors.
2. Multiply the other factor by one of the parts.
3. Then, multiply that product by the other part.

The left example shows that 8 is broken apart into 2×4 . Multiply the first part: $15 \times 2 = 30$. Then multiply the 30 by 4: $30 \times 4 = 120$. So, $15 \times 8 = 120$.

The right example shows that 15 is broken apart into 3×5 . Multiply the first part: $3 \times 8 = 24$. Then multiply the 24 by 5: $24 \times 5 = 120$. So, $15 \times 8 = 120$.

When It's Useful: Breaking apart by factors to multiply is most useful when one factor can be broken apart easily. In the example, breaking apart 8 works well because 2×15 and 30×4 are friendly. Breaking apart 15 into 3×5 works, but 24×5 may be a difficult one to multiply.

Factor Factor

$$15 \times 8$$

$$\begin{array}{c} 2 \times 4 \end{array}$$

$$15 \times 2 = 30$$

$$30 \times 4 = 120$$

$$15 \times 8 = \boxed{120}$$

product

Factor Factor

$$15 \times 8$$

$$\begin{array}{c} 3 \times 5 \end{array}$$

$$5 \times 8 = 40$$

$$40 \times 3 = 120$$

$$15 \times 8 = \boxed{120}$$

product

Break Apart by Addends to Multiply

How It Works: We can break apart either factor.

1. Choose which factor to break apart.
2. Break apart that factor into a sum. Multiply each addend by the other factor.
3. Add all the products together. This gives you the answer to the original problem.

The left example shows that 15 is broken apart into 10 and 5. Multiply the first part, $10 \times 8 = 80$. Then multiply the second part, $5 \times 8 = 40$. The 80 and 40 are added: $80 + 40 = 120$. The answer to the original problem is 120, $15 \times 8 = 120$.

The right example shows that 8 is broken apart into 4 and 4. Multiply the first part: $15 \times 4 = 60$. Then multiply the second part: $15 \times 4 = 60$. The 60 and 60 are added: $60 + 60 = 120$. The answer to the original problem is 120, $15 \times 8 = 120$.

When It's Useful: Breaking apart by addition to multiply is often useful. Any factor can be broken apart in different ways. The goal is to break apart factors in a way that makes the most sense.

Factor Factor

$$15 \times 8$$

$$\begin{array}{c} 10 + 5 \end{array}$$

$$\boxed{10} \times 8 = 80$$

$$\boxed{5} \times 8 = 40$$

$$80 + 40 = 120$$

$$15 \times 8 = \boxed{120}$$

product

Factor Factor

$$15 \times 8$$

$$\begin{array}{c} 4 + 4 \end{array}$$

$$15 \times 4 = 60$$

$$15 \times 4 = 60$$

$$60 + 60 = 120$$

$$15 \times 8 = \boxed{120}$$

product



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

NOTES

TEACHING ACTIVITIES for Break Apart to Multiply

Break Apart to Multiply is one of the first multiplication strategies students learn. It often begins with decomposing basic facts and evolves into multidigit whole numbers. Factors can be broken apart into addends or into factors. In this section, we focus on instructional activities for helping students develop efficient ways to break apart. The goal is that students become adept at decomposition for efficiency and accuracy and also to consider when the strategy is useful.



ACTIVITY 1.1 MULTIPLES OF 10

Students will use their knowledge of basic facts and the associative property to multiply and divide by multiples of 10. Using base-10 blocks is one way to help students see the multiples of 10. First, pose a basic fact expression, such as 3×4 . Briefly discuss the product before posing 3×40 . Provide base-10 blocks so that students can create representations of both expressions.

Multiples of 10			
3×40			
Thousands	Hundreds	Tens	Ones
		□□□□ □□□□ □□□□	
		□□□□ □□□□ □□□□	
		□□□□ □□□□ □□□□	

After they make models of these problems, record the expressions for each problem making the connections explicit. For example, 3×40 can be thought of as 3×4 tens; 40 can be thought of as 4×10 , and so 3×40 is the same as $3 \times 4 \times 10$ or 12×10 (which is also shown with the base-10 blocks).

$$\begin{array}{r} 3 \times 40 \\ / 1 \\ 4 \times 10 \end{array}$$

$$\begin{array}{r} 3 \times 40 \\ 3 \times 4 \times 10 \end{array}$$

$$\begin{array}{r} 3 \times 4 \times 10 \\ 12 \times 10 \\ 120 \end{array}$$

As the numbers get larger, base-10 blocks are not efficient. For example, you would not use base-10 blocks to solve 30×40 . This is why the connections to expressions (as shown) and the deep understanding of breaking apart factors is so important. To solve 30×40 , the factors can be decomposed into $3 \times 10 \times 4 \times 10$ and then rearranged to become $3 \times 4 \times 10 \times 10$ and eventually 12×100 . This work is the underpinning as to why it looks like we are adding zeros when multiplying whole numbers. For instance, 3×4 is 12, which is multiplied by 10 and by 10 again.

(Continued)

(Continued)

30 x 40			
Thousands	Hundreds	Tens	Ones
		1	2
		$\times 10$	$\times 10$
1	2	0	0

There are patterns when multiplying and dividing by multiples of 10. Those have to do with the number of zeros in the product or quotient. Students need to be aware of the patterns, but we mustn't say "just multiply and add zeros." This shortcut, without understanding, is detrimental to work with decimals and leads to many misconceptions.



ACTIVITY 1.2 COLOR TILE ARRAYS

Using color tiles to build arrays and to break them apart is a good way to help students *see* that factors can be broken apart. This is a good introductory activity for students' early learning with multiplication. It is also wise to revisit an activity like this to reinforce a firm understanding of single-digit factors before extending to two- and three-digit factors. First, prompt all partner groups to build an array with 24, 36, 40, 48, or 60 color tiles. After arrays are built, ask students to decompose the arrays into two smaller rectangles and to record their findings. Notice in Deryn's array that the prompt is open so that students come up with a variety of decompositions. You might consider asking students to come up with some or all of the different ways Deryn could have done this. These different possibilities are important for establishing that a factor can be decomposed in many ways.

Deryn made 6×4 array of color tiles.

She broke apart her large array into two smaller arrays.

What were the size of each of the smaller arrays?

6×4

Possible solutions show some of the ways students might break apart the array. Your priority in this activity is to show how the original factors of 6 and 4 are distributed in these examples. Be sure to make explicit connections and ensure that students recognize that the total number of color tiles (24) remains unchanged. Students can then work with arrays of different sizes to see if breaking apart a factor always works.

6×3 6×1

6×2 6×2

4×4
 2×4

3×4
 3×4



ACTIVITY 1.3 BREAKING APART BASIC FACTS

The strategies students learn for recalling basic facts are the groundworks for multiplying multidigit factors. This activity revisits how basic facts can be decomposed.

First, pose a collection of basic facts like those noted and ask students to find the products by breaking apart one or both factors. Note that some students simply know the basic facts, which is good! For this activity, you want them to show how factors can be decomposed so that the concept can be extended.

Basic Facts to Use for Breaking Apart Factors

6×3

8×3

7×6

7×4

7×8

Students should show that any of the factors in these problems might be decomposed. As you know, factors of 3 can be decomposed into $2 + 1$, factors of 6 can be decomposed into $5 + 1$ or 3×2 , factors of 7 can be thought of as $5 + 2$, and factors of 8 can be thought of as $5 + 3$ or $2 \times 2 \times 2$.

After confirming that students recognize and can make use of breaking apart factors, prompt them to think about how they might break apart these problems:

16×3

8×23

7×16

7×24

15×8

Give students time to think about ways that they might decompose these factors. Look for students who continue to break apart the single-digit factor and ask them to think about how they might decompose the two-digit factor instead. It is likely that at least one group of your students will decompose the two-digit factors. However, if this doesn't happen, be sure to acknowledge their thinking, give them an example of how you decomposed a two-digit factor, and then charge them with determining if it can be done.

Note that this early work with decomposing two-digit factors is likely most successful with relatively small, two-digit factors (less than 30). As students show their understanding with these, you can begin to move to two-digit factors greater than 30.



ACTIVITY 1.4 TWO CUTS

Centimeter grid paper is a useful tool for representing multiplication with two-digit factors. In this activity, each student group has a rectangle with the same dimensions. In this example, students have rectangles that are 24×27 cm. Students are tasked with making two cuts so that they create four smaller rectangles. Students should use calculators for this activity.

10×9 90	14×9 126
10×18 180	14×18 252

12×14 168	12×14 168
12×13 156	12×13 156

7×8 56	17×8 136
7×19 133	17×19 323

(Continued)

(Continued)

Students find the area of each smaller rectangle and record the equations. Groups of students will create a variety of examples, as shown here. Bring the class together to discuss the patterns they see within the expressions. Highlight that a factor can be broken apart in any way.

$10 \times 9 = 90$	$12 \times 14 = 168$	$7 \times 8 = 56$
$14 \times 9 = 126$	$12 \times 14 = 168$	$17 \times 8 = 136$
$10 \times 18 = 180$	$12 \times 13 = 156$	$7 \times 19 = 133$
$14 \times 18 = 252$	$12 \times 13 = 156$	$17 \times 19 = 323$

TEACHING TAKEAWAY

Help students align their equations as they record them so the patterns within the factors can be recorded easily.

After discussion, ask students if they think that breaking apart a factor will always work. Then, have them work with rectangles of different sizes to see if it does always work. Consider extending the activity so that students create six rectangles by breaking apart one of the factors into three parts.



ACTIVITY 1.5
THE BETTER BREAK

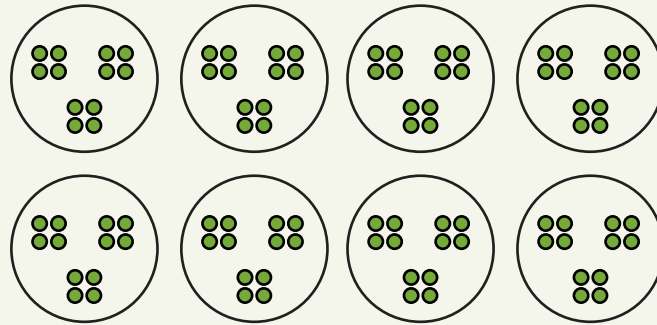
As students show understanding of the Break Apart to Multiply strategy, you want to begin to shift their attention to efficient ways to break apart a factor. After all, 105×13 could be thought of as 21 sets of 5×13 , but that isn't what an efficient, fluent person would do. In this activity, you want students to decompose a problem and then consider different decompositions that you offer to determine which is the better break (apart). For example, you might pose 43×15 . After students break it apart, you might share it broken apart into $(10 + 10 + 10 + 10 + 3) \times (10 + 5)$, $(40 + 3) \times (5 + 5 + 5)$, or $(20 + 20 + 3) \times (10 + 5)$. Then, as a class, discuss which seem to be the better breaks. Keep in mind that efficiency may be different for each student and that accuracy obviously plays a significant role. Because of this, students' early preferences might rely on breaking factors into groups of 10 ($10 + 10 + 10 + 10 + 3$) because they can find products of 10. Some problems for students to work within this activity include the following:

ONE-DIGIT BY TWO-DIGIT	TWO-DIGIT BY TWO-DIGIT	ONE-DIGIT BY THREE-DIGIT	ONE-DIGIT BY FOUR-DIGIT
34×5	54×16	5×670	$2 \times 1,458$
23×8	22×63	353×8	$6 \times 7,080$
55×7	75×13	240×6	$4,125 \times 5$
6×82	29×44	515×8	$3,199 \times 4$
94×4	86×30	925×3	$3,450 \times 6$

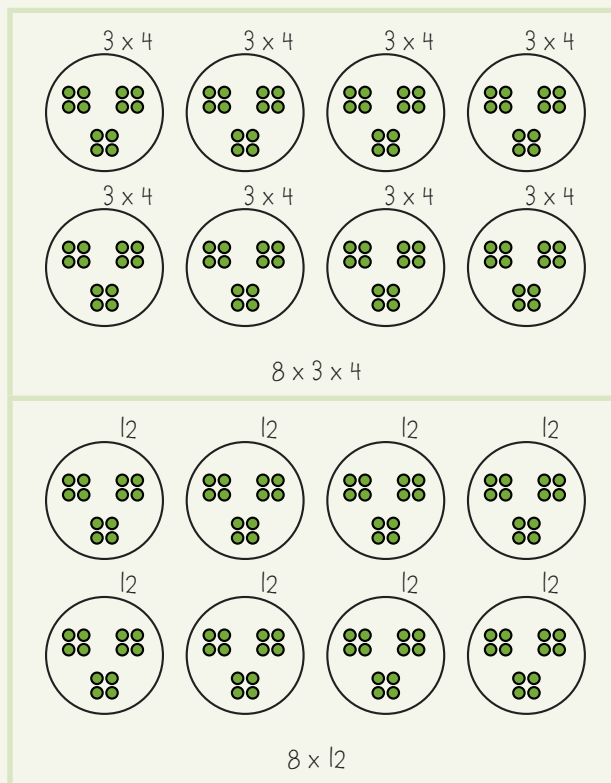


ACTIVITY 1.6 BREAKING APART WITH MULTIPLICATION

Factors can be decomposed by addends or by factors (i.e., with multiplication). This instructional activity focuses on the latter approach. Pose an example of a problem with sets of equal groups. Ask students how they would find the total amount. Give students time to work and then bring the group together to discuss ideas.



After discussion, ask to make an argument for the image showing $8 \times 3 \times 4$ or 8×12 . Student thinking might look something like this:



During discussion, it should become clear to students that $8 \times 3 \times 4$ is the same as 8×12 . You can prompt them to think about why this is so and when it might be useful. You can have them explore the idea further by posing problems and having them find different ways to break apart factors. For example, challenge students to also explain how the visual shows 24×4 and 32×3 . Also ask, “Which of these options would you choose and why?” At first, use representations like those in this example. Good problems include 15×14 , 16×7 , 22×8 , and so on. Each of these is relatively easy to draw. In time, work with problems like 25×16 , 32×14 , or 48×30 , in which students work solely with numeric expressions.



ACTIVITY 1.7 PROMPTS FOR TEACHING BREAK APART TO MULTIPLY

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking, including base-10 models, number lines, arrays, and so on. After students work with the prompt(s), bring the class together to exchange ideas. These could be useful for collecting evidence of student understanding. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once if modified.

- In the expression 15×8 , does it matter which way you break apart the factors? Why or why not? Explain your answer.
- How can you use the Break Apart strategy to solve 8×424 ?
- Jo solved 27×8 by breaking apart 27. How do you think she broke it apart?
- Lee thinks that you can only break apart one factor when multiplying. Do you agree or disagree with Lee? Create a problem to justify your thinking.
- Tyler solved 34×13 . First, he multiplied 30×10 and found 300. What do you think he did next? What else did he have to do to solve the problem?
- Jewel found 16×9 by breaking apart 16 into 4×4 and 9 into 3×3 . She rearranged it to be $4 \times 3 \times 4 \times 3$. Then, she multiplied 12×12 . Does this create the same product? What do you think she could do with 25×12 ?
- Dezi shows that you can multiply 24×40 by breaking 24 into $10 + 10 + 4$ or by breaking 40 into $10 + 10 + 10 + 10$. Do both of Dezi's approaches work? If they do, which do you prefer?
- Create an argument for the best way to break apart 435×7 .
- Jonah thinks that 48×6 is the same as 46×8 because you can break apart the numbers and move them around. Tell why you agree or disagree with Jonah. Use pictures, numbers, or words to show your thinking.
- Tell what you think is the most efficient way to break apart 34×25 .
- Ryan solved 65×33 by breaking apart 33. He did $60 \times 10 + 60 \times 10 + 60 \times 10 + 60 \times 3 + 5 \times 10 + 5 \times 10 + 5 \times 10 + 5 \times 3$. Is there another way you would suggest that he thinks about the problem?
- Create a story problem for 16×45 . Solve it by using the Break Apart strategy.

NOTES

PRACTICE ACTIVITIES for Break Apart to Multiply

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the Break Apart to Multiply strategy or are they reverting to previously learned and/or possibly less appropriate strategies? - Are students using the Break Apart to Multiply strategy efficiently? - Are they breaking apart factors in convenient ways? - Do they break apart factors regardless of appropriateness for the problem at hand? - Do they recognize after carrying out the strategy that there was a more efficient way to break apart the factors?
Flexibility	- Are students carrying out the strategy in flexible ways? (e.g., Do they break apart by addends only? Do they break apart factors in different ways?) - Do they change their approach to how they are breaking apart factors if the results are proving too complicated?
Accuracy	- Are students using the Break Apart to Multiply strategy accurately? (e.g., Do they multiply the correct numbers? Do they add the correct products?) - Are students finding accurate solutions? - Are they considering the reasonableness of their solutions?* - Are students estimating before finding solutions?*

*This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

A strength (and a challenge) of Break Apart to Multiply is that it is flexible. Here are some challenges that students commonly encounter:

- The quantity that is broken apart (e.g., 2) is added back on (rather than adding on two groups).
 - 32×22 : changes to 32×20 to equal 640 and then incorrectly adds 2 to equal 642.
- The student breaks apart a factor but not into parts that they can multiply mentally.
 - 25×48 : breaks apart into $20 \times 48 + 5 \times 48$ and gets stuck.

3. The student thinks of breaking apart using addends (distributive property) or by factors (associative property) but forgets about the other option.
 - 25×48 : breaks apart by addends, forgetting to break apart into factors ($25 \times 4 \times 12$).
4. The student becomes confused about place value when there are zeros in a problem.
 - 305×15 : breaks apart into $30 \times 15 = 450$ and $5 \times 15 = 75$ and gets 525.
5. The student breaks apart as they would for adding two numbers, multiplying tens by tens and ones by ones and omitting the other parts.
 - 34×62 : multiplies $30 \times 60 = 1,800$ and $4 \times 2 = 8$, then adds them to get 1,808.

The prompts from Activity 1.7 can be used for collecting worked examples. Throughout the module are various worked examples that you can use as fictional worked examples. A sampling of additional ideas is provided in the following table.

SAMPLE WORKED EXAMPLES FOR BREAK APART TO MULTIPLY		
	ONE WAY	ANOTHER WAY
Correctly Worked Example (make sense of the strategy) What did _____ do? Why does it work? Is this a good method for this problem?	Patrick's work for 25×48 : $25 \times 48 =$ $25 (40 + 8)$ $25 \times 40 = 1,000$ $25 \times 8 = 200$ $\underline{1,200}$	Theresa's work for 25×48 : $25 \times 4 \times 12$ $\checkmark \quad \swarrow$ $100 \times 12 = 1,200$
Partially Worked Example (implement the strategy accurately) Why did _____ start the problem this way? What does _____ need to do to finish the problem?	Rosi's start for 44×26 : $44 \times 26 = 11 \times 4 \times 26$	Daniel's start for 560×14 : $500 \times 10 = 5,000$ $500 \times 4 = 2,000$
Incorrectly Worked Example (highlight common errors) What did ____ do? What mistake does ____ make? How can this mistake be fixed?	Denisa's work for 708×60 : $70 \times 60 = 4,200$ $8 \times 60 = 480$ $\underline{4,680}$	Christopher's work for 26×46 : $20 \times 40 = 800$ $6 \times 6 = 36$ $800 + 36 = 836$



ACTIVITY 1.8

Name: *“Complex Number Strings”*

Type: *Routine*

About the Routine: Fluency with multiplying two-, three-, and four-digit factors relies on prowess with multiplying multiples of tens, hundreds, and thousands. Students also must be capable beyond just adding zeros. Instead, they must recognize that 7×60 is 7×6 tens and so on. A complex number string helps students practice relationships between basic facts and relationships between basic facts and multiplying multiples.

Materials: This routine does not require any materials.

- Directions:**
1. Provide a matrix of related number strings with one known product. The first row in this routine is always left blank and found first as an anchor for understanding and conversation as needed.
 2. Students use the known product to work across the rows and down the columns.
 3. After students signal that they know the products of each, you hold a class discussion about how the first known relates to the others. Draw students' attention to how basic facts relate to multiplying multiples of tens, hundreds, and thousands. Keep in mind that students will recognize the pattern of zeros in the products. Be sure to reinforce why this pattern makes sense.

$1 \times 1 =$	$10 \times 1 =$	$10 \times 10 =$	$100 \times 1 =$	$100 \times 10 =$
$9 \times 5 = 45$	$90 \times 5 =$	$90 \times 50 =$	$900 \times 5 =$	$900 \times 50 =$
$9 \times 6 =$	$90 \times 6 =$	$90 \times 60 =$	$900 \times 6 =$	$900 \times 60 =$
$9 \times 7 =$	$90 \times 7 =$	$90 \times 70 =$	$900 \times 7 =$	$900 \times 70 =$
$9 \times 8 =$	$90 \times 8 =$	$90 \times 80 =$	$900 \times 8 =$	$900 \times 80 =$
$9 \times 16 =$	$90 \times 16 =$	$90 \times 160 =$	$900 \times 16 =$	$900 \times 160 =$



ACTIVITY 1.9

Name: *“The Breaks”*

Type: *Routine*

About the Routine: Students are most likely comfortable decomposing numbers before they work with multidigit factors. Even so, they may not think to decompose numbers in strategic ways. This routine intends to have students make connections between the different ways factors can be broken apart and then determine which may be most advantageous. This routine can use feature Break Apart by Factors or Addends, but the examples focus on Break Apart by Addends.

Materials: Prepare a collection of related expressions in one row as shown.

- Directions:**
1. Pose a multiplication expression to students, charging them with thinking about how they might break apart the first factor or the second factor.
 2. Students share some ideas with a partner.
 3. Bring the group together to discuss different decompositions. Record the decompositions in two columns, with one column for decompositions of the first factor and one column for decompositions of the second factor. After recording a handful of each, ask students to identify which may be most useful or most friendly. Then, discuss if there are any ideas about how they might break apart both factors for an even better option.

The Breaks: 32×8

BREAK APART 32	BREAK APART 8
$30 \times 8 + 2 \times 8$	$32 \times 4 + 32 \times 4$
$10 \times 8 + 10 \times 8 + 10 \times 8 + 2 \times 8$	$32 \times 5 + 32 \times 2 + 32 \times 1$
$15 \times 8 + 15 \times 8 + 2 \times 8$	$32 \times 2 + 32 \times 2 + 32 \times 2 + 32 \times 2$
$20 \times 8 + 10 \times 8 + 2 \times 8$	

In this example, the teacher posed 32×8 . The teacher recorded student ideas for breaking apart 32 in the first column and ideas for breaking apart 8 in the second column. Many of the students agreed that the shaded expression was the best for them. Some students preferred the first expression ($30 \times 8 + 2 \times 8$). Then, the teacher prompted students to consider if there was a good option for breaking apart both factors. The class agreed that there wasn't for this problem.



ACTIVITY 1.10

Name: *“Strategize Your Break Apart”* Type: *Routine*

About the Routine: Efficiency is part of fluency. A problem can be broken apart in a variety of ways but we want students to think about the most efficient way to do so. This routine helps students consider different ways to break apart to efficiently solve a problem. Note that this routine can also focus on ways to break apart without then solving the problems. This allows for more problems to be explored, although sometimes seeking the answer provides insights into whether the break apart option was, in fact, efficient.

Materials: list of three or four problems of the same operation

- Directions:**
1. Show and discuss a series of problems, one at a time, like a Number Talk (Parrish, 2014).
 2. Ask students to mentally break apart the first expression seen, and signal when ready.
 3. Ask different students to share how they would break apart the expression to solve. Record all ideas.
 4. Ask students to pick one of the Break Apart methods discussed and solve the problem.
 5. Share answers and discuss whether the first Break Apart method mentioned worked out well or not.
 6. Repeat with other problems (as time allows).
 7. Conclude the series with a synthesizing discussion, asking questions such as these:
 - What were efficient ways to break the factors apart?
 - What other ways might you have broken apart the factors?
 - Are there any ways that wouldn't have worked? Are there any ways that wouldn't have worked well?
 - What might be a problem similar to this that wouldn't work well with the break apart strategy?

Examples of problems to use with this routine include the following:

TWO-DIGIT BY ONE-DIGIT	TWO-DIGIT BY TWO-DIGIT	THREE-DIGIT BY ONE-DIGIT	FOUR-DIGIT BY ONE-DIGIT
9×26	12×35	5×125	$7 \times 2,015$
8×42	72×15	8×350	$9 \times 2,700$
6×36	24×25	4×275	$3 \times 4,520$
7×54	16×45	6×120	$4 \times 1,625$



ACTIVITY 1.11

Name: *Factor Race*

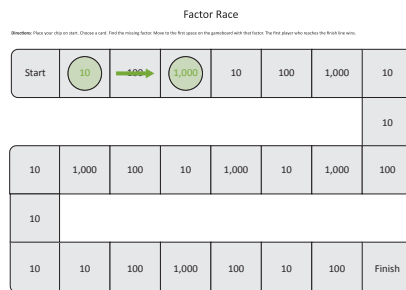
Type: *Game*

About the Game: The skill of multiplying by multiples of 10 is necessary for the Break Apart strategy as well as the other multiplication strategies. *Factor Race* is a practice opportunity for multiplying by multiples of 10, 100, or 1,000.

Materials: *Factor Race* cards and game board, two counters, calculator (one per player)

- Directions:**
1. Stack the *Factor Race* cards face down and place their counter on Start.
 2. Players take turns flipping cards and finding the missing factors.
 3. Players place their counter on top of the next space with a factor that corresponds to their missing factor.
 4. Opponents check for accuracy. If incorrect, the player returns to their previous space.
 5. The player who reaches the Finish block first wins.

In this example, Travis first selected $240 \times \underline{\quad} = 2,400$ and moved his counter to 10. On his second turn, he selected $2 \times \underline{\quad} = 2,000$ and moved his counter to the next 1,000 space.



RESOURCE(S) FOR THIS ACTIVITY

$4 \times \underline{\quad} = 4,000$	$3 \times \underline{\quad} = 30$	$7 \times \underline{\quad} = 700$
$6 \times \underline{\quad} = 60$	$9 \times \underline{\quad} = 900$	$2 \times \underline{\quad} = 2,000$
$8 \times \underline{\quad} = 800$	$5 \times \underline{\quad} = 50$	$2 \times \underline{\quad} = 200$



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.12

Name : *Mult-zee*

Type: *Game*

About the Game: *Mult-zee* is a take on the classic dice game, *Yahtzee*. It is an engaging way to practice multiplication using the Break Apart to Multiply strategy. *Mult-zee* can be modified in a variety of ways. As you change the game, you will need to change the conditions. For example, if you were to feature 2 two-digit factors, you would want to change “product is greater than 500” to “product is greater than 5,000” or “product is less than 500” to “product is less than 5,000.”

Materials: 10-sided dice, four decks of digit cards (0–9), or playing cards (queens = 0, aces = 1, kings and jacks removed); *Mult-zee* game board

- Directions:**
1. Players deal themselves three digit cards to make a two-digit and a one-digit factor.
 2. Players find the product of the factors.
 3. Players use the product to satisfy one of the conditions on the game board by recording it in the corresponding space.
 4. After each turn, the player must choose one of the places on the game board for the product they found or take a zero for that condition.
 5. After 11 turns, the players add their scores and the highest score wins.

Note that there are two “chance” spaces on the game board. In these spaces, the product is the point value for the chance space.

RESOURCE(S) FOR THIS ACTIVITY

Mult-zee

Directions: Make a multiplication problem with digit cards. Find the product. Choose a space to record your product. You only have 11 turns.

Player 1		Player 2	
Points	My Problem	Points	My Problem
Even Product	100	Even Product	100
Odd Product	100	Odd Product	100
Product with a 5 in it	100	Product with a 5 in it	100
Product with a 3 in it	100	Product with a 3 in it	100
Product has two digits that are the same	500	Product has two digits that are the same	500
Product is greater than 500	100	Product is greater than 500	100
Product is less than 500	100	Product is less than 500	100
Product is between 500 and 750	200	Product is between 500 and 750	200
Chance (Product is your score)		Chance (Product is your score)	
Chance (Product is your score)		Chance (Product is your score)	
Product is 1,000	1,000	Product is 1,000	1,000
Total score		Total score	



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.13

Name: *Break Apart Bingo*

Type: *Game*

About the Game: *Break Apart Bingo* helps students think about how they might break apart factors when multiplying.

Materials: *Break Apart Bingo* boards (one per student), counters, and expression cards (or list of expressions)

- Directions:**
1. Players take turns pulling expression cards.
 2. Players solve the expression by using the Break Apart to Multiply strategy.
 3. Then, players place a counter on decomposed parts they used to solve the problem.
 4. Bingo is five in a row in any direction.

For example, the expression card 78×4 was called. Player 1 (left game board) broke apart 78 into 70 and 8. She covered the 8 because there was no 70 on her game board. Player 2 (right game board) broke it apart in the same way and was able to cover both numbers. Note that a player might have thought about 78 as $50 + 20 + 8$ and could potentially cover three spaces. It is important that students first solve the problem before looking at their spaces, so they don't create inefficient decompositions to win the game.

Break Apart Bingo

Directions: Use the numbers at the top to fill in the boxes on the game board. You can write a number more than once.

2	3	4	5	6	7	8	9	20	30	40	50	60	70	80	90
2	30	20	50	4											
9	8	60	20	5											
40	20	4	40	7											
60	50	5	30	4											
3	80	20	9	8											

Break Apart Bingo

Directions: Use the numbers at the top to fill in the boxes on the game board. You can write a number more than once.

2	3	4	5	6	7	8	9	20	30	40	50	60	70	80	90
5	60	70	8	3											
90	40	6	5	20											
3	80	20	9	8											
5	6	50	40	3											
20	7	30	8	4											

You can find a sample list of expressions for playing the game on the companion website, although it will work with any examples. Expressions can also be written on index cards if you want small groups of students to play *Bingo* without you.

Blank 5×5 game boards can be copied or downloaded. Students can then create their own boards by writing the numbers 2–9 and multiples of 10 (20–90) anywhere on their board. Note that students can break apart factors by 1 and 10 but that might challenge the performance of the game. If a student uses 8 tens on his board and the first problem is 83×6 , he could cover all 8 of his tens.



Game cards and game boards can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.14

Name: *Break Apart to Multiply Math Libs*

Type: *Game*

About the Game: Games do not always have to be ripe with strategy. Students can enjoy rather simplistic games of chance. *Break Apart to Multiply Math Libs* is a race to complete the equations. It can be adjusted for use with any strategy.

Materials: *Break Apart to Multiply Math Libs* game board; digit cards (0–9), playing cards (queens = 0, aces = 1, kings and jacks removed), or a spinner

- Directions:**
1. Players take turns pulling a digit card and using the number to fill a space on their board.
 2. Once a player fully completes a problem, they use the Break Apart to Multiply strategy to find the unknown. Their opponent also finds the unknown to confirm they are correct (but cannot record it on their game board).
 3. A player loses their turn if a digit they pull can't be used for any space on their board.
 4. The first player to find all of the unknowns correctly wins.

For example, on Jackson's first turn he pulled a 5 and used it to fill the third 5×10 in the third row. Then, he pulled a 1 and completed the first row, finding the product. On his third turn, he pulled a 4 and completed the third row, finding the product of 170.

$21 \times 6 = ?$	$(20 \times 6) + (\underline{1} \times 6) = ?$	$? = 126$
$45 \times 7 = ?$	$(20 \times \underline{\quad}) + (20 \times \underline{\quad}) + (5 \times 7) = ?$	$? =$
$5 \times 34 = ?$	$(5 \times 10) + (5 \times 10) + (\underline{5} \times 10) + (5 \times \underline{4}) = ?$	$? = 170$

RESOURCE(S) FOR THIS ACTIVITY

Break Apart to Multiply Math Libs		
Directions: Take turns choosing a digit card. Use the digit to fill a number in the break apart column. Once a problem is completely filled in, find the product. Be the first to find all of the products.		
$21 \times 6 = ?$	$(20 \times 6) + (\underline{\quad} \times 6) = ?$	$? =$
$45 \times 7 = ?$	$(20 \times \underline{\quad}) + (20 \times \underline{\quad}) + (5 \times 7) = ?$	$? =$
$5 \times 34 = ?$	$(5 \times 10) + (5 \times 10) + (\underline{\quad} \times 10) + (5 \times \underline{\quad}) = ?$	$? =$
$9 \times 22 = ?$	$(\underline{\quad} \times 20) + (9 \times \underline{\quad}) = ?$	$? =$
$13 \times 8 = ?$	$(10 \times 8) + (\underline{\quad} \times 8) = ?$	$? =$
$55 \times 6 = ?$	$(55 \times \underline{\quad}) \times 3$	$? =$
$3 \times 72 = ?$	$(2 \times \underline{\quad} 2) + 72$	$? =$



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.15

Name: *Same But Different*

Type: *Center*

About the Center: This center helps students recognize that numbers can be broken apart in different ways when multiplying. This strategy develops efficiency as students compare various ways to decompose a factor.

Materials: Same But Different cards, recording sheet

- Directions:**
1. Students select a Same But Different card.
 2. Students copy the first Break Apart expression and solve.
 3. Students copy the second Break Apart expression and solve.
 4. After working with the center, students should be asked to explain how the problems are the same and how they are different.

In the following example, a student picks up the card 8×15 . The student solves side A. Side A shows that the 8 was broken apart into $4 \times 15 + 4 \times 15$, which equals 120. Then the student solves side B. This card is missing a factor, so the student would have to think about which of the factors was broken apart. In this case, it was the 8. The students would then complete the expression and solve: $2 \times 15 + 6 \times 15 = 120$.

8×15	
A	B
4×15 4×15	2×15 $__ \times 15$

RESOURCE(S) FOR THIS ACTIVITY

Same But Different Cards			
8×15		5×44	
A	B	A	B
4×15 4×15	2×15 $__ \times 15$	1×44 4×44	5×40 5×4
12×30		424×8	
A	B	A	B
2×30 10×30	6×30 6×30	2×424 2×424 4×424	4×424 4×424



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.16

Name: **Backward Breaks**

Type: **Center**

About the Center: Reversing an activity is a good way to enrich a task (SanGiovanni, Katt, & Dykema, 2020). In this center, students are given a decomposed multiplication problem and they have to find the product and recompose it.

Materials: Backward Breaks cards, recording sheet

- Directions:**
1. Students choose a card and record the information.
 2. Students find the product and record it.
 3. Then, students determine the original problem.
 4. Students determine if the decomposed expression card was the most efficient way (for them) to think about the problem or they write another way that the problem might have been solved.

Decomposed expression cards, including the examples here, are available for download.

RESOURCE(S) FOR THIS ACTIVITY

$5 \times 70 + 5 \times 1$	$4 \times 50 + 4 \times 20 + 4 \times 3$
$20 \times 6 + 20 \times 6 + 6 \times 6$	$20 \times 9 + 20 \times 9 + 1 \times 9$
$30 \times 5 + 30 \times 5 + 5 \times 5 + 2 \times 5$	$40 \times 60 + 3 \times 60 + 40 \times 4 + 3 \times 5$

Backward Breaks			
Directions: Select a decomposed expression card. Find the product. Write the original problem. Tell if there is a more efficient way to think about the problem.			
Decomposed Problem	Product	Original Problem	Is there a more efficient way to think about the problem?
$6 \times 30 = 6 \times 20 + 6 \times 9$	$6 \times 30 = 60, 6 \times 20 = 120, 6 \times 9 = 54$ $60 + 120 + 54 = 234$ So, the product is 234.	6×29	It would be more efficient to think about 6×20 and then add 6×9 .



Center cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.17

Name: *Break Apart With an Area Model Representation*

Type: *Center*

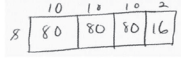
About the Center: Early work with the Break Apart to Multiply strategy, and multidigit multiplication in general, uses precise area models. In time, instruction shifts to a more generic representation of an area model and eventually to a rather simplistic, abstract representation in which the parts are similar in size. This center is an opportunity for students to practice using this generic, area model representation and connecting it to the related equations.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed); Break Apart With an Area Model Representation recording sheet; generic area model recording sheet

- Directions:**
1. Students pull digits to create a multiplication problem relative to the size of factors they have been working with. For example, they pull three digit cards to make a two-digit by one-digit problem or four digit cards to make a two-digit by two-digit problem.
 2. Students record the multiplication expression.
 3. Students break apart one or both factors and represent their thinking with a generic area model.
 4. Students record the equations represented by the area model.

Although this center activity seems rather simplistic, it offers a window into students' thinking. They are not directed how to break apart the factors, so you will see if they rely on groups of 10 as in the example or if they work with groups of tens. You can extend the center by asking students to create a problem and show two different ways to break apart the factors, or you can ask students to tell how they selected the break aparts that they did. This center also has potential as an assessment artifact.

RESOURCE(S) FOR THIS ACTIVITY

Breaking Apart		
Directions: Use digit cards to create a multiplication problem. Break apart the factors to show how you multiplied. Show your thinking with an area model. Show your thinking with numbers.		
Problem	Show It With an Area Model	Show It With Numbers
32×8		$10 \times 8 = 80$ $10 \times 8 = 80$ $10 \times 8 = 80$ $2 \times 8 = 16$ 256

1	2	3	4
5	6	7	8
9	1	2	3
4	5	6	7
8	9	0	0



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 1.18

Name: *Which and How?*

Type: *Center*

About the Center: As students become comfortable with the Break Apart to Multiply strategy, you want them to begin to think strategically about which factor to decompose and how to decompose it. This center offers an opportunity to practice choosing which factor to decompose and how it might be decomposed.

Materials: multiplication expression cards, math journal or recording sheet

- Directions:**
1. Students select a multiplication expression card and record it.
 2. They circle the factor they choose to decompose.
 3. Then, students record how they decomposed the factor and find the product.

In this example, the student first selected the problem 36×22 . She decided to break apart 22 into $10 + 10 + 1 + 1$ and multiplied each by 36 finding a product of 792.

Problem	How
$36 \times \textcircled{22}$	$36 \times 10 + 36 \times 10 + 36 \times 1 + 36 \times 1$ $360 + 360 + 36 + 36$ $720 + 72$ 792
$\textcircled{14} \times 53$	$10 \times 50 + 10 \times 3 + 2 \times 53 + 2 \times 53$ $500 + 30 + 106 + 106$ $530 + 212$ 742

Problem	How
$15 \times \textcircled{32}$	15×32 $\swarrow \searrow$ $15 \times 4 \times 8$ 60×8 480
$\textcircled{55} \times 20$	55×20 $11 \times 5 \times 20$ 11×100 $1,100$



A multiplication expression card template can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

Halve and Double Strategy

STRATEGY OVERVIEW:

Halve and Double

What is Halve and Double? As the name implies, this strategy involves changing two factors, halving one of them and doubling the other. It is a useful strategy for multiplication facts. For example, 5s facts can be thought of as halving the other number and doubling 5:

$$5 \times 8 = 10 \times 4$$

$$5 \times 9 = 10 \times 4.5$$

The strategy also generalizes to larger numbers:

$$5 \times 18 = 10 \times 9$$

$$50 \times 48 = 100 \times 24$$

You can also halve and double more than once if continuing to use the strategy makes the problem easier to multiply:

$$25 \times 640 = 50 \times 320$$

$$50 \times 320 = 100 \times 160$$

$$= 16,000$$

A key to this strategy is being adept at halving numbers and doubling numbers. In other words, students need automaticity with halving and doubling.

HOW DOES HALVE AND DOUBLE WORK?

This strategy is a special case of the Break Apart Into Factors where one factor is broken apart into $2 \times$ [the rest of the factor]. That 2 is associated with the factor, doubling it. It looks like this:

$$8 \times 35 = (4 \times 2) \times 35$$

$$= 4 \times (2 \times 35)$$

$$= 4 \times 70$$

$$= 280$$

WHEN DO YOU CHOOSE THE HALVE AND DOUBLE STRATEGY?

This strategy is very useful any time one factor is 5, 50, 500, and so on. When the other factor is even, it is all the more convenient, but the other factor does not have to be even for the strategy to work.

$$28 \times 5 = 14 \times 10$$

$$34 \times 50 = 17 \times 100$$

$$79 \times 500 = 39.5 \times 1,000$$

Doubling a number that ends in a 5, like 35, results in a tens number, like 70. This often makes the adapted problem easier to multiply, even if it continues to be a written problem and not a mental problem:

$$68 \times 350 = 34 \times 700$$

HALVE AND DOUBLE: Strategy Brief for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. This strategy brief is a tool for doing that. You can include it in parent or school newsletters or share it at parent conferences. It is available for download so that you can adjust it as needed.

Halve and Double

How It Works: We can halve a factor and double another to get an answer.

1. Choose which factor to halve and which to double.
2. Halve one factor.
3. Double the other factor.
4. Find the product by multiplying.

The left example shows that you can halve 14 and double 15. These new numbers make it easier to multiply. Now the number sentence is 7×30 , which is 210. So, $14 \times 15 = 210$.

The right example shows a different way that you can halve and double. The problem is 25×8 : 8 can be halved and 25 doubled to create 50×4 . You can double and halve again: 4 can be halved and 50 doubled, resulting in 100×2 . $100 \times 2 = 200$ is the same as $25 \times 8 = 200$.

When It's Useful: Halving and doubling is useful when one factor is an even number. It also helps when both factors can be easily halved and doubled. For example, 38×17 is not a good fit for this strategy because 19×34 is not a friendly problem to solve.

$$14 \times 15$$

Halve → 7 Double → 30

$$7 \times 30 = 210$$

$$25 \times 8$$

Double → 50 Halve → 4

$$50 \times 4 = 200$$

$$100 \times 2 = 200$$



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

MODULE

1

Break Apart to
Multiply
Strategy

MODULE

2

Halve and
Double
Strategy

MODULE

3

Compensation
Strategy

MODULE

4

Partial
Products
Strategy

MODULE

5

Think
Multiplication
Strategy

MODULE

6

Partial
Quotients
Strategy

MODULE

7

Standard Algorithms
for Multiplication
and Division

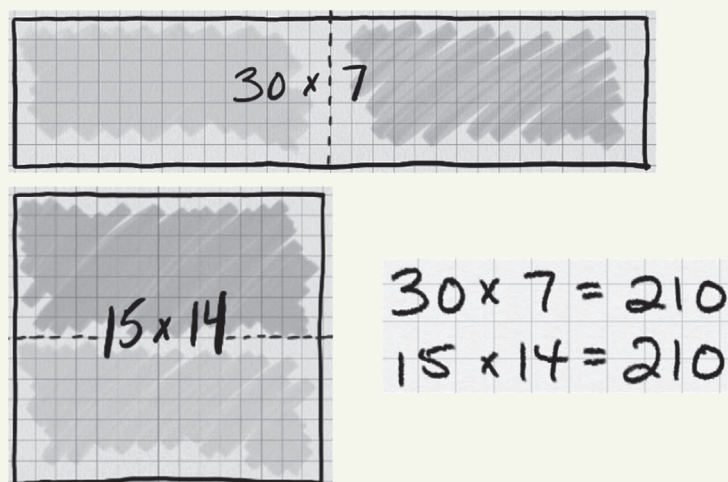
TEACHING ACTIVITIES for Halve and Double

Before students are able to choose strategies, a key to fluency, they first must be able to understand and use relevant strategies. These activities focus specifically on Halve and Double. Halve and Double is a strategy that works well in certain cases. While students may employ other methods, which is appropriate, they also must learn this strategy to develop their flexibility and efficiency. Students must understand that halving a factor while doubling the other yields the same result. You can develop this concept by working with a variety of representations and connecting them to equations. In this section, we focus on instructional activities for helping students develop understanding of the Halve and Double strategy and for determining when it is appropriate. Fluency with this strategy will build as students develop skill with halving or doubling numbers in general.



ACTIVITY 2.1 HALVING AND REARRANGING RECTANGLES

In this activity, you pose rectangles to students for them to trace on grid paper and cut out. Rectangle dimensions to use could include 15×14 , 28×5 , 16×6 , 25×22 , and 16×16 . Students record the equation for the area of one of their rectangles. Then, they cut the rectangle in half and rearrange it to make one, longer rectangle and record the equation for it. In the following figure, the original rectangle was 15×14 . The student cut it in half, creating two rectangles that were each 15×7 . Then, she aligned those two smaller rectangles, creating a rectangle that was 30×7 . She recorded the equations for each rectangle's area.



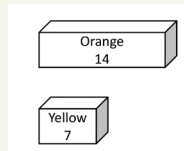
Students repeat halving and rearranging for the other rectangles that you pose. As students work, you want them to think about how the factors and the area of the corresponding rectangles are similar and different. After students complete the activity, a group discussion should be had about how one factor is halved and how the other is doubled yet the product (area) remains the same. Ask students to determine if this idea always works and then have them find examples to prove their case. It's possible that students will overlook that one factor is always even. If so, be sure to insert a rectangle, like 13×15 or 17×21 , for them to explore.

Even though this is an introduction to the concept of halving and doubling, it isn't too soon to discuss when this technique might be a good idea. Notice that 15×14 and 28×5 from the suggested set create expressions that yield multiples of 10 (30×7 and 14×10 , respectively). Because of this, the Halve and Double strategy is a good choice for these expressions.

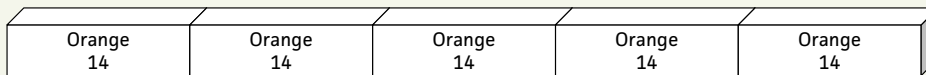


ACTIVITY 2.2 CUISENAIRE ROD PROOFS

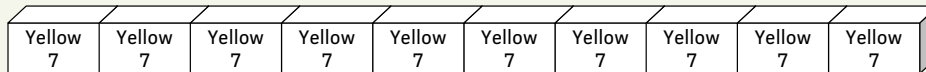
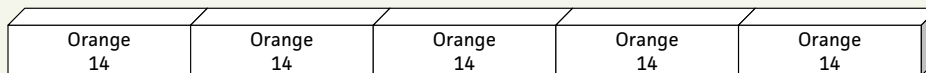
Cuisenaire rods are useful for developing a variety of concepts beyond fractions. In this activity, students work with orange and yellow Cuisenaire rods to investigate halving and doubling. First, be sure that students recognize that a yellow rod is exactly half of an orange rod. Then, you can create a train of oranges with an assigned value. For example, you might begin with orange having a value of 14 and ask students what the value of a yellow rod would be (7).



Then, ask students to build a train of five orange rods and have them determine how long the train is. They will find that a train of five orange rods is 70.



Then, ask students to predict and find how many yellow rods it would take to create a train of yellow rods with the same length of 70.

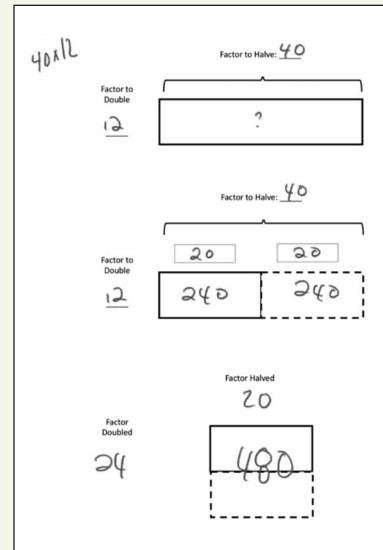
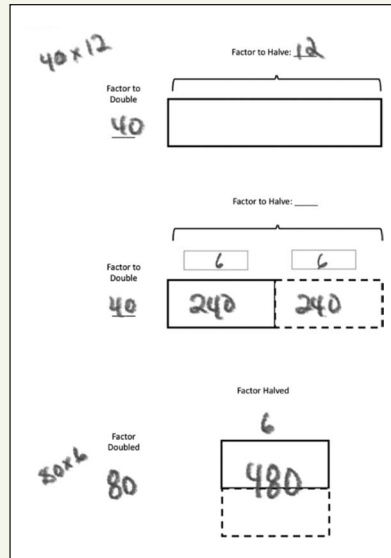


Be sure to have students record equations that describe the trains that they make. Some students may need a calculator to support their accuracy, which is fine. In this example, the two equations to record are $5 \times 14 = 70$ (orange train) and $10 \times 7 = 70$ (yellow train). You can repeat the activity endlessly by asking, “What if the orange had a value of ____?” You can also change the number of orange rods in a train. For example, you could pose that an orange rod has a value of 22 and there are 7 of them, yielding a length with the value of 154 (22×7). Then, students would work to find a yellow train with 14 rods and the same length of 154. Keep in mind that you want to look for patterns in the equations, how they represent the strategy of halving and doubling, and what values (factors) this strategy works best with.



ACTIVITY 2.3 REPRESENTING HALVE AND DOUBLE WITH GRAPHIC ORGANIZERS

Graphic organizers can be a useful tool for understanding the concept of a strategy and how it works. Graphic organizers, like the following Halve and Double organizer, work well when joined with other instructional activities such as Activities 2.1 or 2.2. This organizer has students identify which factor to halve and which to double. The middle section focuses the action on halving and the resulting products. The bottom section shows how the halved product can be repositioned to create the new problem. In the examples, the student works with 40×12 . She halves 12 into 6 and 6 and eventually 80×6 or 480. The examples show the possible rich conversations that can come from these activities. In it, the student halves 40 instead of 12 and their subsequent work is shown. That rich conversation then comes from the questions you might ask: Which factor do you think was a better choice for halving? How does it help getting to a related problem that connects with basic facts? Why do some problems let us halve and double either factor?



This graphic organizer can be downloaded at resources.corwin.com/figuringoutfluency.



ACTIVITY 2.4 NICKELS AND DIMES

Nickels and dimes may be one of the most effective contexts for developing the halving and doubling strategy. They also represent a situation in which a problem can be thought of with multiples of 10. The activity is a rather straightforward problem. You tell students that Bag A has a certain amount of nickels and students work to find how many dimes would be in another bag (Bag B) so that the two bags have the same amount of money. Here are two problems that students solved.

Bag A has 28 nickels.

$$28 \times 5 = 140 \text{¢}$$

Bag B has 14 dimes.

$$14 \times 10 = 140 \text{¢}$$

28×5 is the same as 14×10

Bag A has 44 nickels.

$$44 \times 5 = 220 \text{¢}$$

Bag B has 22 dimes.

$$22 \times 10 = 220 \text{¢}$$

44×5 is the same as 22×10

Students created the following table to show the number of nickels and dimes and to show how they recorded the corresponding equations after each problem. The table and equations help students see how the factors are halved and doubled.

n	d	
16	8	$16 \times 5 = 8 \times 10$
22	11	$22 \times 5 = 11 \times 10$
28	14	$28 \times 5 = 14 \times 10$
32	16	$32 \times 5 = 16 \times 10$
44	22	$44 \times 5 = 22 \times 10$

This activity is also quite useful for showing how the halving and doubling strategy can work with odd numbers. For example, if you have 7 nickels you can think of this as 3 dimes and a nickel (35¢) or you can think of it as 3.5 dimes, and $3.5 \times 10 = 35$.

This activity can be adapted to exploring quarters (25¢). For example 40 quarters (40×25) is the same as 20×50 or 10×100 , to equal 1,000¢ or \$10.



ACTIVITY 2.5 PROMPTS FOR TEACHING HALVE AND DOUBLE

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking, including base-10 models, number lines, arrays, and so on. After students work with the prompt(s), bring the class together to exchange ideas. These could be useful for collecting evidence of student understanding. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once if modified.

- David stated that 18×6 is the same as 9×3 . Kayce disagreed and said that 18×6 is the same as 9×12 . How do we know that Kayce is correct? What was David's mistake?
- Sara said the halve and double strategy works with division too, but it is halve and halve instead (or double and double). For example, $240 \div 6$ is the same as $120 \div 3$. Justify whether or not this is always true.
- Ciera said that doubling and halving always works. Is that correct? If not, give an example of when it doesn't work.
- Is Halve and Double the most effective strategy when solving for 20×15 ? Why or why not?
- Does halving and doubling always work well? Create problems to justify your thinking.
- Che says you can't use Halve and Double for 15×17 . Maria says she can use the strategy. She says she just has to make one of the factors even. She gives 1 from 17 to 15, creating 16×16 . What do you think about Maria's strategy?
- Russ says 4×15 is perfect for using the Halve and Double strategy but that 12×15 and 9×15 aren't good examples of when to use the strategy. Explain why you agree or disagree with Russ.
- Write your own multiplication word problems that could be solved using the Halve and Double strategy.
- Michael used the Halve and Double strategy to solve 8×15 . He halved 8 and doubled 15. Michael came up with 4×30 and found a product of 120. Would the strategy still be efficient if Michael halved 15 and doubled 8 instead?
- Create a story problem for 5×22 . Solve using Halve and Double.

NOTES

PRACTICE ACTIVITIES for Halve and Double

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the Halve and Double strategy or do they rely on other strategies regardless for how appropriate they are for the problem at hand? - Are students using the Halve and Double strategy efficiently? - Do they use the strategy regardless of its appropriateness for the problem at hand?
Flexibility	- Are students carrying out the strategy in flexible ways? (e.g., With this strategy do they make certain tens well and others not as well?) - Do they change their strategy to Halve and Double as they realize it will be more useful? (e.g., When solving 35×8, a student begins to break apart and then realizes they can make the problem 70×4.) - Do they change their approach from Halve and Double as it proves too complicated for the problem at hand?
Accuracy	- Are students using the Halve and Double strategy accurately? (e.g., Do they halve and double or double without halving?) - Are students finding accurate solutions? - Are they considering the reasonableness of their solutions?* - Are students estimating before finding solutions?*

*This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

Halve and Double is useful in special cases, so a major challenge is to just remember it *is* a strategy and be on the lookout for when it will work. Other challenges include the following:

1. The student makes an error in halving an odd number.
 - 65×5 : changes to 38.5×10 .
2. The student has trouble choosing which number to halve and which to double.
 - 450×12 : changes to 225×24 , rather than 900×6 .

A key idea is to look for a number that, when doubled, is more manageable to multiply. In addition, noticing a second opportunity to use Halve and Double can continue to make the problem easier to work. Worked examples can attend to these ideas. The prompts from Activity 2.5 can be used for collecting worked examples. Throughout the module are various worked examples that you can use as fictional worked examples. A sampling of additional ideas is provided in the table.

SAMPLE WORKED EXAMPLES FOR HALVE AND DOUBLE

Correctly Worked Example (make sense of the strategy) What did _____ do? Why does it work? Is this a good method for this problem?	Trevor's work for 250×28: 250×28 500×14 $1,000 \times 7 = 7,000$
Partially Worked Example (implement the strategy accurately) Why did _____ start the problem this way? What does _____ need to do to finish the problem?	Neesha's start for 15×260: $30 \times 130 =$ Dante's start for 25×68: $50 \times 34 =$
Incorrectly Worked Example (highlight common errors) What did _____ do? What mistake does _____ make? How can this mistake be fixed?	Greta's work for 50×45: $100 \times 23.5 = 2,350$



ACTIVITY 2.6

Name: *“A String of Halves”*

Type: *Routine*

About the Routine: Making use of the Halve and Double strategy relies on students’ ability to double and find halves. Often, students do well with doubling but finding a half proves more challenging. “A String of Halves” aims to help students improve their skill with halving. This routine uses a set of numbers to help students see how numbers are halved.

Materials: Prepare a set of numbers that are intentionally related.

- Directions:**
1. Post a set of three or more related numbers as shown.
 2. Ask students to mentally find the halves of as many numbers as possible.
 3. If students are unable to find all of the halves, have them talk with partners about how they can use the relationships between numbers and the known halves to find the unknown half. If students find all of the halves, have them discuss with partners how the numbers and their halves are related. To extend this situation (when all halves are known), have students generate a new number and its half that is related to the set.

4.	^A			^B		
	40	16	56	30	40	50
	40	16	56	30	40	50
	20	8		20	25	

These two examples of “A String of Halves” were used in a fourth-grade classroom. The teacher started with Example A, recording 40, 16, and 56 because she wanted to develop an idea about how to halve 56. Students were asked to think of the half for each of the numbers. They could halve 40 and 16. At that point, she asked students to talk with partners about how they could figure out the half of 56 by knowing the halves of 40 and 16. The teacher then used Example B in a similar way. She posed the three related numbers, and students found the halves. She then asked students to look for patterns and relationships between the numbers and the halves to prove what half of 30 is.

Halving numbers like 76 or even 90 can be challenging for students because there is an element of regrouping within the half. This routine can be leveraged for developing skill with those halves by making use of relationships and patterns. In the following figure, four multiples of 10 (60, 70, 80, and 90) are posed. Students are tasked with halving each, but the conversation focuses first on 60 and 80 before students work to find half of 70 and 90. During these discussions, focus their attention on half of 10 being added to or taken from half of 60 or 80 and why. Of course, another strategy is to think of the half (35) between those halves (30 and 40).

^C			
60	70	80	90
60	70	80	90
30		40	



ACTIVITY 2.7

Name: *“The Same As”*

Type: *Routine*

About the Routine: This routine helps students think about how to use the Halve and Double strategy. As students become comfortable with the strategy, you might sprinkle in expressions that don’t call for the strategy (6×50) or those that prove problematic for the strategy (7×13).

Materials: Prepare two or three expressions for students to restate.

- Directions:**
1. Pose a few expressions to students.
 2. Students talk to each other about another way they could think about the problem by either doubling or halving. To note, students do not need to solve the expressions, just rewrite them.
 3. Students discuss which of the expressions would be easier to solve.

THE SAME AS		THE SAME AS	
6×12 is the same as _____	35×8 is the same as _____	350×6 is the same as _____	16×450 is the same as _____

These are two different examples of the routine. The left example shows two-digit by one-digit problems. First, students were tasked with rethinking 6×12 . Students share that it can be thought of as the same as 3×24 . The second prompt, 35×8 , is the same as 70×4 . After students shared suggestions for these two, they then discussed which situation was a good use of the strategy (35×8 because it becomes a friendly problem). The right example shows how the routine can be used with larger factors. In this case, both problems are good for the Halve and Double strategy.



ACTIVITY 2.8

Name: *The Splits*

Type: *Game*

About the Game: Skill with halving is useful in many ways and necessary for leveraging the Halve and Double strategy. *The Splits* is a practice opportunity for halving. The object of the game is to generate numbers, halve them, and put the halves in order from least to greatest.

Materials: place value dice, digit cards (0–9), or playing cards (queens = 0, aces = 1, kings and jacks removed); *The Splits* game board

- Directions:**
1. Players take turns generating a two-digit number. Note that three- and four-digit numbers are also options.
 2. Players halve the number and record the result in one of the five boxes.
 3. Each subsequent half must be placed in a box so that halves are in order from least to greatest.
 4. If a half cannot be placed, the player loses their turn. (Optional: If an odd number is generated, the player loses their turn also.)
 5. The first player to fill their five boxes wins.

For example, students are playing *The Splits* by finding half of two-digit numbers. Toby's first number is 44. He halves it and places 22 in the second box. After his partner's turn, Toby's second number is 70. Half of 70 is 35 and he places it to the right of 22. Toby's third number is 62. He loses his turn because there is no space to place the resulting half, 31.

Player 1

	22	35		
--	----	----	--	--

TEACHING TAKEAWAY

Games are great homework assignments. Families can also use math games to keep kids busy, like when waiting for dinner at a restaurant.

RESOURCE(S) FOR THIS ACTIVITY

The Splits

Directions: Take turns rolling dice or pulling cards to make a number. Find half of that number. Record the half in one of the spaces. The halves you put in the boxes have to be in order from least to greatest. If a number doesn't fit because of other numbers you have already played, you lose your turn. Be the first to fill all of your boxes.

--	--	--	--	--



Digit cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 2.9

Name: *Get 10*

Type: *Game*

About the Game: *Get 10* is a quick-and-easy game for practicing finding halves. The goal of the game is to be the first player to correctly halve 10 numbers.

Materials: 19 bingo chips or counters, four decks of digit cards or playing cards with faces and tens removed

- Directions:**
1. Players take turns pulling two digit cards.
 2. Players have the chance to earn two points, finding half of each two-digit number that can be formed with their cards.
 3. Opponents confirm accuracy with a calculator, and the player who correctly halved their numbers claims two chips.
 4. First player to get 10 chips wins. For example, Chala draws a 5 and a 2; he says half of 52 is 26 and half of 25 is 12.5 or 12 and a half. He is correct both times and gets two chips.

This game can be played with three- and four- digit numbers, but rather than have multiple options, the player organizes the three or four cards in any order and finds half of that number.

TEACHING TAKEAWAY

Calculators ensure accurate practice. For this game, you might need to train students how to find half on a calculator.



ACTIVITY 2.10

Name: *Halve and Double Flips*

Type: *Game*

About the Game: *Halve and Double Flips* practices identifying multiplication problems that can be solved with the Halve and Double strategy. Players place chips on corresponding spaces and can flip opponents' pieces that lie directly between two of their pieces (similar to *Othello*).

Materials: *Halve and Double Flips* game cards and game board, multiplication expression cards, and two-colored counters for game pieces

- Directions:**
1. Players take turns pulling a multiplication expression card.
 2. Players consider if the expression can be solved more efficiently using the Halve and Double strategy.
 3. If the expression can be solved more efficiently with the strategy, the players put their counter on any space labeled "WORKS" (i.e., the Halve and Double strategy works for this expression).
 4. If the expression cannot be solved more efficiently with the strategy, the player puts their counter on any space labeled "DOESN'T WORK" (i.e., the Halve and Double strategy doesn't make the problem easier to solve).
 5. When placing a game piece, any opponent pieces between a previously placed piece and the current piece are flipped (similar to *Othello*). Note that unlike *Othello*, a player does not have to place a piece at the end of a row of pieces.
 6. The game ends when all of the spaces are filled.
 7. The player with the most chips on the board wins the game.

RESOURCE(S) FOR THIS ACTIVITY

44×5	14×15
3×16	50×18
40×28	32×6
16×6	18×40
28×5	5×48

Halve and Double Flips							
Directions: Pull an expression card. Choose a space that tells if you can or can't use the Halve and Double strategy with the expression. After placing a piece, the user may capture pieces caught between your new piece and a piece you already had on the board. The player with the most pieces when the board is filled wins.							
Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK
Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS
Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double DOESN'T WORK	Halve and Double WORKS
Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS
Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double DOESN'T WORK	Halve and Double WORKS
Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS	Halve and Double DOESN'T WORK	Halve and Double WORKS



Game cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 2.11

Name: *The First to Seven* Type: *Game*

About the Game: *The First to Seven* is a game for considering expressions that might be best solved with the Halve and Double strategy. Players attempt to get 7 points by collecting seven expression cards that can be efficiently solved with the Halve and Double strategy. After each successful flip, students must decide if they want to keep going or pass to their opponent.

Materials: four decks of digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed)

- Directions:**
1. Player 1 begins by using digit cards to make a multiplication problem.
 2. Player 1 determines if the problem can be solved with the Halve and Double strategy and renames it accordingly and solves it mentally. If they solve it correctly, they score a point and have the choice of going again or passing.
 3. If player 1 continues playing, they can earn another point by creating a second problem that can be solved with the Halve and Double strategy. If the player continues playing and cannot mentally solve the problem using Halve and Double, the player loses all of the points they have earned.
 4. A player only goes back to zero if they attempt to get consecutive points.
 5. The first player to 7 points wins the game.

TEACHING TAKEAWAY

During the game, have students record Halve and Double expressions. After the game, have students determine which expressions are truly best for the strategy.

For example, player 1 turns up 2, 4, 5, 1 and forms the expression 24×15 . They say aloud, “I can halve 24 and double 15. That is 12 times 30, which is 360.” Their opponent checks the answer on a calculator. Player 1 draws again. They draw 3, 6, 1, and 0. They form 30×16 , and explain that they can halve and double to get 60×8 , which is 480. They decide to stop, and play goes to their opponent.



ACTIVITY 2.12

Name: *Fives and Tens*

Type: *Center*

About the Center: This center is designed as an independent practice intended to focus on situations where multiples of tens are useful applications of the Halve and Double strategy. Often, that applies to leveraging multiples of 10. This center might pair well with Activity 2.4 (“Nickels and Dimes”). Fives and tens can be modified to focus students’ attention on a similar relationship between 15 and 30, 25 and 50, or any other multiple of 5 and corresponding multiple of 10 as shown.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), 10-sided dice, and Fives and Tens recording sheets

- Directions:**
1. Students generate a number by drawing two cards or rolling the dice two times..
 2. Even numbers can be recorded as a factor in the $\times 5$ column or the $\times 10$ column.
 3. Odd numbers are recorded in the $\times 10$ column.
 4. After placing the number as a factor in one of the columns, students write the related $\times 5$ or $\times 10$ problem that is generated by the Halve and Double strategy.

RESOURCE(S) FOR THIS ACTIVITY

5s and 10s			
Directions: Generate a number. Use even numbers as a factor for the $\times 5$ or $\times 10$ column and use odd numbers for the $\times 10$ column only. Write the related multiplication problem and find the product.			
Number I Made	$\times 5$ Column	$\times 10$ Column	Product of Both
16	16×5	8×10	80
	$___ \times 5$	$___ \times 10$	

15s and 30s			
Directions: Generate a number. Use even numbers as a factor for the $\times 15$ or $\times 30$ column and use odd numbers for the $\times 30$ column only. Write the related multiplication problem and find the product.			
Number I Made	$\times 15$ Column	$\times 30$ Column	Product of Both
16	16×15	8×30	240
	$___ \times 15$	$___ \times 30$	

25s and 50s			
Directions: Generate a number. Use even numbers as a factor for the $\times 25$ or $\times 50$ column and use odd numbers for the $\times 50$ column only. Write the related multiplication problem and find the product.			
Number I Made	$\times 25$ Column	$\times 50$ Column	Product of Both
16	16×25	8×50	400
	$___ \times 25$	$___ \times 50$	

35s and 70s			
Directions: Generate a number. Use even numbers as a factor for the $\times 35$ or $\times 70$ column and use odd numbers for the $\times 70$ column only. Write the related multiplication problem and find the product.			
Number I Made	$\times 35$ Column	$\times 70$ Column	Product of Both
16	16×35	8×70	560
	$___ \times 35$	$___ \times 70$	



Digit cards and recording sheets can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 2.13

Name: *Write It as Halve and Double*

Type: *Center*

About the Center: This center prompts students to think about how they can rethink a multiplication problem using the Halve and Double strategy. It also asks them to think about if the strategy is useful for the problem. Students select an expression card, rewrite it with the strategy, and tell if using the strategy is efficient for the problem.

TEACHING TAKEAWAY

Like other centers, this one can be modified for other strategies by renaming expressions with the Compensation or Think Multiplication strategies.

Materials: Write It as Halve and Double cards and recording sheet

- Directions:**
1. Students pull an expression card.
 2. They rewrite the expression using the Halve and Double strategy.
 3. They tell if the Halve and Double strategy is useful for the problem.

RESOURCE(S) FOR THIS ACTIVITY

Write It as Halve and Double		
Directions: Select a multiplication problem card. Rewrite it using the Halve and Double strategy. Tell if using the strategy is a good choice for the problem.		
Problem on the multiplication card	Rewrite the problem using the Halve and Double strategy	Is the strategy a good choice for the problem? Why?

40×4	45×4
16×5	32×5
22×5	5×48
38×5	5×32
5×42	7×45



Game cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

This image shows a blank sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There is a green header bar at the top of the page. The paper appears to be a standard notebook or ledger page.

Compensation Strategy

STRATEGY OVERVIEW:

Compensation

What is Compensation? Compensation involves making a change or adjustment to one of the factors and then compensating for that change to maintain equivalence. Often, Compensation involves rounding up one of the factors, multiplying, then compensating for the extra. For example, with basic facts, the 9s facts can be developed by thinking of 9×6 as 10×6 and then subtracting one group of 6 (compensating for the extra group of 6). This also extends to larger numbers:

$$\begin{aligned} 99 \times 6 &= 100 \times 6 - 6 \\ &= 600 - 6 \\ &= 594 \end{aligned}$$

HOW DOES COMPENSATION WORK?

Compensation employs the distributive property:

$$\begin{aligned} 12 \times 48 &= 12(50 - 2) \\ &= 12 \times 50 - 12 \times 2 \\ &= 600 - 24 \\ &= 576 \end{aligned}$$

A key is to keep track of the number of *groups* (or sets or rows or columns) added and subtract the same number. In this example, 48 was adjusted to 50 (adding two extra columns of 12). In the end, those two extra groups (24) are subtracted to compensate.

WHEN DO YOU CHOOSE COMPENSATION?

This strategy is very useful when one of the factors is close to a benchmark number like 70 or 50 or 200 and that benchmark can be mentally multiplied by the other factor.

$$68 \times 7 = 70 \times 7 - 14$$

$$8 \times 49 = 50 \times 8 - 8$$

$$198 \times 15 = 200 \times 15 - 30$$

Notice that in these three examples, Break Apart to Multiply is also a reasonable option. However, because the ones place is a larger digit (i.e., is close to the next benchmark), Compensation is likely easier. Compensation can be applied to other benchmark numbers, such as 25. For example, 6×24 can be solved with Compensation by thinking $6 \times 25 = 150$ and then subtracting a group of 6.

COMPENSATION: Strategy Brief for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. This strategy brief is a tool for doing that. You can include it in parent or school newsletters or share it at parent conferences. It is available for download so that you can adjust it as needed.

Compensation

How It Works: We can change a factor to find a product and then you can compensate for the change.

1. Decide on a friendlier computation.
2. Change a factor and find the product.
3. Compensate (undo) the amount that was changed.

The left example shows that there are 49 groups of 7: 49×7 . You can change 49 to 50 because 50 is easier to multiply with. This means you are adding an extra group to make it 50 groups instead of 49. Multiply 50 groups of 7: $50 \times 7 = 350$. Then, you can take away the extra group of 7: $350 - 7 = 343$.

The right example shows that 23×8 can be changed to 25×8 , which means you added 2 groups to 23. You now have 25 groups: $25 \times 8 = 200$. Then, take away the two extra groups of 8 (i.e., 16): $200 - 16 = 184$. So, $23 \times 8 = 184$.

When It's Useful: Compensation is useful any time a factor can be easily changed to a factor that is easier to multiply. For example, 23×17 can be thought of as $23 \times 20 - 23 \times 3$, but that number sentence may not be easier to multiply.

Factor	Factor
49	7
49×7	
(50-1)	
$50 \times 7 = 350$	
$350 - 7 = 343$	
product	

Factor	Factor
23	8
23×8	
(25-2)	
$25 \times 8 = 200$	
$200 - 16 = 184$	
product	



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

MODULE
1

Break Apart to
Multiply
Strategy

MODULE
2

Halve and
Double
Strategy

MODULE
3

Compensation
Strategy

MODULE
4

Partial
Products
Strategy

MODULE
5

Think
Multiplication
Strategy

MODULE
6

Partial
Quotients
Strategy

MODULE
7

Standard Algorithms
for Multiplication
and Division

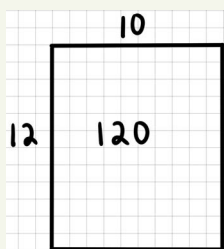
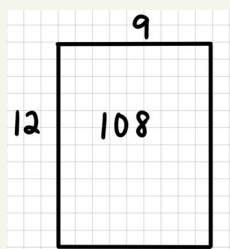
TEACHING ACTIVITIES for Compensation

Before students are able to choose strategies, a key to fluency, they first must be able to understand and use relevant strategies. These activities focus specifically on Compensation. Compensation is a strategy that works well in certain cases. While students may employ other methods, which is appropriate, they also must learn this strategy to develop their flexibility and efficiency. Students must understand how expressions that have common factors are related and how a number of groups must be taken away or added on when expressions are adjusted. You can develop this concept by working with a variety of representations and connecting them to equations. In this section, we focus on instructional activities for helping students develop understanding of the Compensation strategy. Compensation with multiplication is different from compensation with other operations, so we also offer activities to expose this difference.



ACTIVITY 3.1 SHOW ME

In this activity, students compare the area of related expressions to prove that compensation works. At first, you might use grid paper with relatively small factors so that students can find the exact area of each. Pose two expressions with a common factor such 12×9 and 12×10 . Have students use grid paper to make two rectangles with those dimensions and find the area of both. You might consider having students cut out each rectangle to see how they are related when one is placed on top of the other. Have students discuss how the dimensions of the rectangles are similar and how they are different.



$$12 \times 9 = 108$$
$$12 \times 10 = 120$$

After discussion about the first set of rectangles, prompt students to repeat the activity with pairs of different rectangles, including 15×9 and 15×10 , 20×9 and 20×10 , and 25×9 and 25×10 . Before students work with each set, have them discuss how the factors are related and how the expressions are similar to the first investigation. Bring the class together after they have found the area of each rectangle described above. Record their findings as shown in the following figure, so that patterns might be observed by students.

$$15 \times 10 = 150 \quad 20 \times 10 = 200 \quad 25 \times 10 = 250$$
$$15 \times 9 = 135 \quad 20 \times 9 = 180 \quad 25 \times 9 = 225$$

Again, ask students to think about how the expressions are the same and different. Extend the activity by asking them to create sets of expressions that show a similar relationship. The next day you can repeat the activity with expressions that multiply by 8 and by 10.

As you move to developing the Compensation strategy with 2 two-digit factors, you can have students use open area representations. For example, you might have them work with 12×30 and 12×28 to determine how the products are related.

$$\begin{array}{r}
 30 \\
 12 \overline{) 360} \\
 \underline{360} \\
 0
 \end{array}$$

$$\begin{array}{r}
 28 \quad 2 \\
 12 \overline{) 24} \\
 \underline{24} \\
 0
 \end{array}$$

$$360 - 24 = 336$$

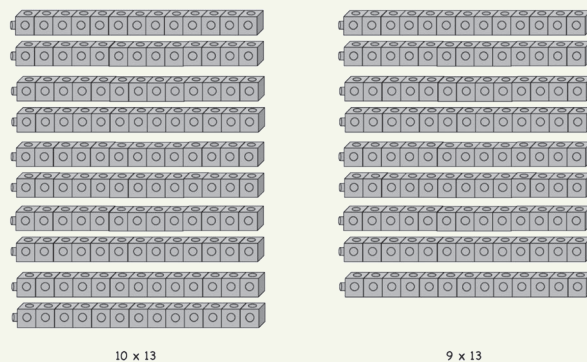
$$30 \times 12 \quad 2 \times 12 \quad 28 \times 12$$

As with the earlier expressions, help students see how the products of two expressions with common factors are related. In this case, 30×12 is two groups of 12 more than 28×12 .

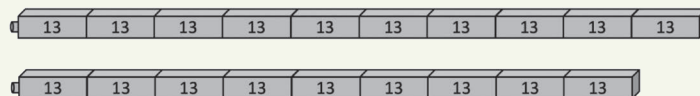


ACTIVITY 3.2 COMPENSATED TRAINS

Area models work well to show how products compare when the Compensation strategy is used. Students often benefit from a different representation to galvanize their understanding of a concept. This activity functions similarly to Activity 3.1 in that you want to have students compare the results of two related expressions. Also know that this physical model loses functionality as the size of the factors increases. Have students create a train of linking cubes of a certain size such as 13. Then, have students predict and find how many cubes would be in 9 trains of 13 cubes and how many cubes would be in 10 trains of 13 cubes. As with Activity 3.1, be sure to record the equations for each and have students examine the equations ($10 \times 13 = 130$ and $9 \times 13 = 117$).



Alternatively, you can have each student in your class build a train of 13 and then have 10 students link their trains and compare its overall length with a train built by 9 students.





ACTIVITY 3.3 TRUE OR FALSE COMPENSATION STATEMENTS

Students may believe that Compensation for multiplication functions similarly to Compensation with addition. For example, they may believe that 38×8 is the same as $40 \times 8 - 2$ because they added 2 to 38, rather than recognize they are adding 2 groups of 8. In this activity, you provide students with a collection of statements and they have to work to prove if they are **true or false**. Students should be asked to justify their thinking. After working with the activity, the class should be brought back together to debrief and solidify that when you use Compensation for multiplication, you are adjusting with groups.

Here are some possible prompts for two-digit by one-digit multiplication:

True or False:

- To solve 9×15 , I can add 1 to 9 to get 10×15 (150) and compensate by subtracting 1.
- To solve 58×5 , I can add 2 to 58. Then 60×5 is 300, and I subtract 2 fives to equal 290.
- To solve 9×11 , I go up 1 for one, down 1 for the other to get 10×10 . My answer is 100.

Here are some examples of prompts for other multidigit multiplication situations:

True or False:

- To solve 19×25 , I multiply 25 times 20, which is 500, and subtract 25 to equal 475.
- 175×8 is the same as $175 \times 10 - 2$.
- 25×38 is the same as 25×40 , and then subtract 50.
- 35×99 is the same as 35×100 , then subtract 1.

For more examples see Activity 3.13.



ACTIVITY 3.4 NUMBER STRINGS

Number strings are a set of related equations that are organized intentionally so that patterns and relationships can be easily observed and used. Compensation number strings are a good tool for helping students make sense of the strategy. First, prepare a string for students to work with. Gather students together and present one equation at a time. Students solve each new problem and share how they solved the problem. During discussion, be sure that you help establish or reinforce the connections between consecutive equations. Here are some questions you might ask: How are the equations similar and different? How are the first two equations related? What patterns do you notice in the equations? Why do you think that pattern is so? What do you think would come next? How can you use what you discovered to help you?

STRING A	STRING B	STRING C	STRING D
$4 \times 10 = ?$	$4 \times 10 = ?$	$6 \times 10 = ?$	$7 \times 10 = ?$
$4 \times 9 = ?$	$4 \times 9 = ?$	$6 \times 8 = ?$	$7 \times 7 = ?$
$14 \times 9 = ?$	$4 \times 19 = ?$	$16 \times 8 = ?$	$17 \times 7 = ?$
$24 \times 9 = ?$	$4 \times 29 = ?$	$26 \times 8 = ?$	$27 \times 7 = ?$
$34 \times 9 = ?$	$4 \times 49 = ?$	$36 \times 8 = ?$	$37 \times 7 = ?$
$44 \times 9 = ?$	$4 \times 69 = ?$	$46 \times 8 = ?$	$47 \times 7 = ?$

STRING E	STRING F	STRING G	STRING H
$4 \times 10 = ?$	$45 \times 1 = ?$	$4 \times 8 = ?$	$7 \times 3 = ?$
$4 \times 4 = ?$	$45 \times 10 = ?$	$6 \times 8 = ?$	$70 \times 3 = ?$
$40 \times 40 = ?$	$45 \times 20 = ?$	$60 \times 80 = ?$	$70 \times 30 = ?$
$40 \times 39 = ?$	$45 \times 40 = ?$	$4 \times 80 = ?$	$70 \times 300 = ?$
$40 \times 38 = ?$	$45 \times 80 = ?$	$64 \times 80 = ?$	$69 \times 300 = ?$
	$45 \times 79 = ?$	$64 \times 79 = ?$	$69 \times 290 = ?$

Number strings can be challenging to create at first. A good tip for creating them is to first think about the problem you want to build toward. In the third example of the second row, we thought about 64×79 . Then, we considered that we would use 64×80 and put it right above our end result. We then thought that 64×80 could be found by using 60×80 and 4×80 and positioned them before 64×80 . Finally, 60×80 and 4×80 naturally connected to the basic facts, so we started with those problems.



ACTIVITY 3.5 PROMPTS FOR TEACHING COMPENSATION

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking, including base-10 models, number lines, arrays, and so on. After students work with the prompt(s), bring the class together to exchange ideas. These could be useful for collecting evidence of student understanding. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once if modified.

- How can you use Compensation to find the product of 49×40 ?
- Chloe uses Compensation for 9×7 and 19×7 . How do you think she uses Compensation for these two problems?
- Create examples of multiplication problems that are good for using the Compensation strategy.
- Jimmy says that anytime a factor has a 9 in it, you should use the Compensation strategy. Do you agree or disagree with Jimmy? Use examples to justify your thinking.
- Jade says that Compensation only works with a two-digit and a one-digit factor like 39×8 . Do you agree or disagree? Use examples to justify your thinking.
- Heidi solved 23×17 by thinking 20×20 and giving 3 from 23 to 17. Do you agree with Heidi's thinking?
- Samuel used Compensation to solve 3×296 . He first changed the 296 to 300. Then he multiplied 3×300 , which gave him a product of 900. Now Samuel does not know what to do. What should Samuel do? Explain your answer.
- Why are compatible numbers helpful when using the Compensation strategy?
- Nia used Compensation to solve 9×21 . She changed the expression to 9×20 , then added a group of 9. Tyler used Compensation to solve 9×21 . He changed the expression to 10×21 , then subtracted a group of 21. Both students had the same product. How are Nia and Tyler's work similar? How are they different?
- Hines wonders if he has to use Compensation for 8×61 by making it 10×61 . Would you suggest he use Compensation or would you suggest something different? Explain your thinking.
- Create a story problem for 8×65 . Solve using Compensation.

PRACTICE ACTIVITIES for Compensation

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the Compensation strategy or are they reverting to previously learned and/or possibly less appropriate strategies? - Are students using the Compensation strategy efficiently? - Do they use the strategy regardless of its appropriateness for the problem at hand?
Flexibility	- Are students carrying out the strategy in flexible ways? (e.g., Do they use Compensation by adding on groups or taking them away? Do they compensate for one group [13×9] and two groups [13×8]?) - Do they change their approach to or from Compensation as it proves inappropriate or overly complicated for the problem?
Accuracy	- Are students using the Compensation strategy accurately? (e.g., Do they remove one or one group 13 when compensating with 13×9?) - Are students finding accurate solutions? - Are they considering the reasonableness of their solutions?* - Are students estimating before finding solutions?*

*This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

A strength (and a challenge) of Compensation is that it is flexible. As with Break Apart to Multiply, the most common errors are as follows:

- The student compensates by a quantity (e.g., 2) rather than groups of that quantity (e.g., two groups).
 - 14×18 : changes to 14×20 , multiplies to equal 280, and then subtracts 2 (rather than two groups of 14) to incorrectly get 278.
- The student adjusts the problem but does not compensate for that change.
 - 499×4 : changes problem to 500×4 , multiplies to equal 2,000, and stops.

Sometimes both factors are good candidates for adjusting. Comparing both options through worked examples can help students gain insights into how to use the strategy. The prompts from Activity 3.5 can be used for collecting worked examples. Throughout the module are various worked examples that you can use as fictional worked examples. A sampling of additional ideas is provided in the following table.

SAMPLE WORKED EXAMPLES FOR COMPENSATION	
Correctly Worked Example (make sense of the strategy) What did ____ do? Why does it work? Is this a good method for this problem?	James' work for 398×14 : $\begin{array}{r} 400 \times 14 = 5,600 \\ 2 \times 14 = \quad 28 \\ \hline 5,572 \end{array}$
Partially Worked Example (implement the strategy accurately) Why did ____ start the problem this way? What does ____ need to do to finish the problem?	Kanish's start for 49×9 : 49×10 Keera's start for 49×9 : 50×9
Incorrectly Worked Example (highlight common errors) What did ____ do? What mistake does ____ make? How can this mistake be fixed?	Greta's work for $2,995 \times 8$: $\begin{array}{l} 3,000 \times 8 = 24,000 \\ 24,000 - 5 = 23,995 \end{array}$



ACTIVITY 3.6

Name: *“A Little More or A Little Less”*

Type: *Routine*

About the Routine: Compensation can be applied in different ways. This routine helps students consider different ways to adjust a problem in order to solve it efficiently.

Materials: Prepare three or four expressions that are good for the Compensation strategy.

Directions: 1. Pose a few expressions to students.

2. Give students time to think and talk to each other about ways they could adjust the problem by either adding on a little more or taking a little less to create a new expression. Students do not need to solve the expressions, but instead just rethink them.

3. Then, talk with the whole group about how problems were adjusted, which approaches were most efficient, and why.

TEACHING TAKEAWAY

Trying to find a solution can distract students from making sense of the process. You can alleviate that by not asking for an answer.

A LITTLE MORE OR A LITTLE LESS

9×21

39×9

8×28

For example, 9×21 is one of three expressions presented in the routine. One student suggests that a little less is 9×20 . Another student suggests that a little more is 10×21 . You record the students' suggestions. At this point, the class discusses which is the easiest to think about and why. Be careful to avoid suggesting that there is always a better or best way. Some expressions to use may include the following:

8×31	4×42	5×51	3×62	2×73	8×398
4×82	2×94	9×19	8×29	7×39	3×698
5×49	3×58	2×67	4×78	2×86	4×199
4×201	3×704	8×402	9×501	5×602	7×299



ACTIVITY 3.7

Name: *“Or You Could”*

Type: *Routine*

About the Routine: As students encounter multiplication problems, they often begin to work on the computation before thinking about another, more efficient approach to the problem. And, within the compensation strategy, students need experience selecting the number they will change and considering how to compensate for that change. This routine asks students to consider how they might think of the problem in a friendlier way.

Materials: Prepare two or three expressions for students to work with.

- Directions:**
1. Pose two or three expressions and ask students to consider how they might think of them for a friendlier computation.
 2. Students think to themselves and then share with partners about an alternative.
 3. Bring the class together to discuss how students thought about the problem.

For example, you might pose 48×4 to a fourth-grade class. Students might share that you could do 48×4 or you could do 50×4 and take 8 away. Another student might share that you could break apart 48×4 , finding $10 \times 4 + 10 \times 4 + 10 \times 4 + 10 \times 4 + 8 \times 4$. Or you could break 48×4 into $24 \times 4 + 24 \times 4$ or even $40 \times 4 + 8 \times 4$. After these ideas are collected, the conversation then focuses on which approaches are easiest to carry out and which are most efficient. Because this routine is situated here, you may choose to limit the experience to applying the Compensation strategy to the expressions you pose.

This routine works with any problem. Some expressions to begin with include the following:

39×6	498×6	4×135	25×48
7×78	399×7	$7 \times 2,178$	39×15
5×27	219×3	$5 \times 2,527$	99×34
9×18	5×496	$9 \times 3,618$	22×88
48×4	2×799	$1,248 \times 4$	19×46
39×4	4×898	$4,639 \times 4$	98×51
8×35	297×4	$8 \times 4,035$	69×77
87×5	198×9	$7,587 \times 5$	49×81



ACTIVITY 3.8

Name: *Compensation Points*

Type: *Game*

About the Game: Students need ample opportunities to practice with the Compensation strategy. *Compensation Points* is an engaging way to practice Compensation and to reason about how factors influence products.

Materials: 10-sided dice, digit cards (0–9), or playing cards (queens = 0, aces = 1, kings and jacks removed); *Compensation Points* game board for each student; calculator

- Directions:**
1. Players take turns generating two digits to complete a line on their game board.
 2. The player places the two digits on a line item and uses Compensation to find the product.
 3. Players check their opponents' accuracy with a calculator. Inaccurate products cannot be used as a score.
 4. After all lines are complete, players add up their products to determine their score.

2 digit number	×	1-digit number	How I Compensated	Product
21	×	9	$20 \times 9 = 180$ $180 + 9 = 189$	189

In this example, the player pulls a 2 and a 9, making 21×9 (note that the “1” was prefilled). The player uses Compensation accurately, finding a product of 189.

RESOURCE(S) FOR THIS ACTIVITY

Compensation Points

Directions: Roll a die the two times or pick up two digit cards. Fill in the empty boxes to create a two-digit number (x) and a one-digit number. Write down how you compensated, then solve and write your product. After the final round, combine all of your products. Compare your score with your partner's. The highest score wins.

2-Digit Number	×	1-Digit Number	How I Compensated	Product
1	×			
9	×			
2	×			
8	×			
7	×			

Player: Total Score:



Digit cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 3.9

Name: *Give Some to Get Some*

Type: *Game*

About the Game: *Give Some to Get Some* focuses students' attention on an amount they would give to one factor and the related extra quantity that needs to be compensated in the end. It helps students focus on how many groups of a factor are added and then removed.

Materials: one *Give Some to Get Some* game board, one regular six-sided die per pair of students

- Directions:**
1. Players take turns rolling the die to generate a number.
 2. The player uses that number to look for an expression in which the amount rolled could be used to adjust one of the factors to make a friendlier problem. For example, a player rolls a 2. She selects 68×30 because she could rethink the problem as 70×30 and then take two groups of 30 (60) away. She places her counter on 68×30 .
 3. If a player rolls a 4, 5, or 6, they lose their turn because factors adjusted by these amounts do not typically yield an efficient problem.
 4. The game ends when all of the spaces are filled.
 5. The player who has the most three-in-a-rows wins the game.

TEACHING TAKEAWAY

Hold students accountable by having them record their play. A player might record that 68×30 is the same as $70 \times 30 - 60$.

RESOURCE(S) FOR THIS ACTIVITY

Give Some to Get Some				
Directions: Take turns rolling a number. Choose a space on the gameboard where the amount you roll could be given from one factor to the other to show the Compensation strategy. The game ends when all spaces are covered. The player with the most three-in-a-rows wins.				
99×98	38×12	39×16	79×24	16×80
37×69	17×25	38×63	48×23	89×26
97×5	88×17	47×54	89×27	25×38
68×30	59×13	68×29	77×17	99×58
27×44	57×43	68×27	46×27	35×46



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 3.10

Name: *Compensation Concentration*

Type: *Game*

About the Game: *Compensation Concentration* helps students think about how the Compensation strategy can be applied to multiplication problems, creating new, friendly expressions. It is played similarly to a traditional game of *Memory* or *Concentration*.

Materials: one deck of *Compensation Concentration* cards

- Directions:**
1. Players place cards face down in an array.
 2. Players take turns flipping over cards, attempting to match an expression with a card that shows how to solve the problem with partial products.
 3. The player who makes the most matches wins the game.

RESOURCE(S) FOR THIS ACTIVITY

28×9	$28 \times 10 = 280$ and take 28 away
8×45	$10 \times 45 = 450$ and take 90 away
8×41	$8 \times 40 = 320$ and add 8
12×9	$12 \times 10 = 120$ and take away 12
49×7	$50 \times 7 = 350$ and take away 7



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 3.11

Name: *Estimation Tic-Tac-Toe*

Type: *Game*

About the Game: *Estimation Tic Tac Toe* practices estimating products of problems that might be solved with the Compensation strategy.

Materials: *Estimation Tic-Tac-Toe* game board and game cards

- Directions:**
1. Players choose estimation cards and place their mark (X or O) on an expression that is a good estimate for their card.
 2. The first player to get three in a row wins.

For example, player 1 (X) selects the card with 19×38 , uses 20×40 as an expression for estimating the product, and places their X on the board. Player 2 (O) selects the card with 29×20 and uses 20×30 as their estimate.

20×20	20×30	20×40
20×50	20×60	20×70
20×80	20×90	20×100

RESOURCE(S) FOR THIS ACTIVITY

Estimation Tic-Tac-Toe

Directions: Player 1 is X and Player 2 is O. Pick up a card. Choose a space that would be a good estimation for the expression picked. Place your mark on that space. The first player with three marks in a row wins.

2×20	2×30	2×40
2×50	2×60	2×70
2×80	2×90	2×100

3×20	3×30	3×40
3×50	3×60	3×70
3×80	3×90	3×100

4×20	4×30	4×40
4×50	4×60	4×70
4×80	4×90	4×100

5×20	5×30	5×40
5×50	5×60	5×70
5×80	5×90	5×100

2×18	2×72
2×27	2×83
2×36	2×89
2×45	2×98
2×61	2×11



Game cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 3.12

Name: *Creating Compensations*

Type: *Center*

About the Center: Countless problems can be adjusted in very similar ways and even by the same amount. For example, 6×229 and 6×239 are both likely to be adjusted by the same amount of 1. However, students can develop a misconception that only certain problems can be adjusted by 1, 2, or 3. This center is a high-order thinking activity for students to create problems that could be adjusted by the given amount.

Materials: six-sided die, Creating Compensations recording sheet

- Directions:**
1. Students generate a number by rolling the die.
 2. Students use the number to generate an expression that they would solve by adjusting one of the factors by the amount they generated.
 3. Students capture their thinking on the recording sheet.
 4. Note that Compensation always works but it is not always efficient. When a student rolls a 4, 5, or 6, they should still create a problem that can be compensated by that amount. However, you may add a direction for students to identify if the compensation is an efficient way to think of the problem.

For example, a student rolls a 3. She creates the expression 4×497 because she can adjust 497 by 3, creating 4×500 . She describes that adjustment in her writing, noting that she would have to take away 12 because she adjusted by four groups of 3.

RESOURCE(S) FOR THIS ACTIVITY

Creating Compensations		
Directions: Roll a number. Create a problem that you would adjust by that amount to solve. Record your thinking.		
Number I Pulled	Problem I Created	How I Would Adjust It by the Number I Pulled
3	4×497	I can think of 4×500 and then take away 12.



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 3.13

Name: *Prove It*

Type: *Center*

About the Center: Students can confuse how compensation works in different situations. This center helps them reinforce or refine their reasoning.

Materials: Compensation Prove It cards and recording sheet

Directions: 1. A student selects a Compensation Prove It card.
2. The student explains that the card is true or false and records their thinking on the recording sheet.

For example, a student pulls a card that says 4×47 is the same as 4×50 and take away 12. The student determines that the statement is true, since 4×3 was added to 4×47 to make it 4×50 . The product for 4×50 is 200. Then we have to remove 4×3 from 200. Note that students don't have to find products necessarily to prove their thinking.

RESOURCE(S) FOR THIS ACTIVITY

A 59×3 is the same as 60×3 and take away 3	A 59×3 is the same as 60×3 and take away 3
B 6×28 is the same as 6×30 and take away 2	B 6×28 is the same as 6×30 and take away 2
C 5×68 is the same as 5×70 and take away 10	C 5×68 is the same as 5×70 and take away 10

Prove It	
Directions: Pull a Prove It card. Prove if it is true or false	
Card I Pulled	Proof That It Is TRUE or FALSE



Game cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 3.14

Name: *Use This for Solving*

Type: *Center*

About the Center: Two very different expressions can be adjusted to the same expression. This center provides a given equation and students determine what other problems it can be used to solve. The example offered is for multiplying two-digit factors but can be easily modified.

Materials: Use This for Solving center cards and recording sheet

- Directions:**
1. Students select a Use This for Solving equation like $40 \times 82 = 3,280$.
 2. Students then generate two different expressions that could be adjusted to this problem to be solved. For $40 \times 82 = 3,280$, the student might create 39×82 and 41×82 .
 3. Students then use the known equation to help them find the product of the two created problems: $39 \times 82 = 3,198$ because it has one less group of 82 and $41 \times 82 = 3,362$ because it has one more group of 82.

RESOURCE(S) FOR THIS ACTIVITY

Use This for Solving		
Directions: Pull an equation. Create and solve two problems related to the equation.		
Equation	I Can Use This for Solving	I Can Use This for Solving
$40 \times 82 = 3,280$	39×82 $3,280 - 82 = 3,198$ $39 \times 82 = 3,198$	41×82 $3,280 + 82 = 3,362$ $41 \times 82 = 3,362$

$20 \times 53 = 1,060$	$40 \times 66 = 2,640$
$50 \times 32 = 1,600$	$42 \times 60 = 2,520$
$60 \times 25 = 1,500$	$35 \times 80 = 2,800$
$40 \times 34 = 1,360$	$62 \times 50 = 3,100$
$35 \times 60 = 2,100$	$40 \times 75 = 3,000$



Center cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

[illegible]

Partial Products Strategy

STRATEGY OVERVIEW:

Partial Products

What is Partial Products? Partial Products breaks numbers apart (usually by place value), and then makes sure each part of one factor is multiplied by each of the other parts. Break Apart to Multiply (by addends) and Partial Products are essentially two names for the same idea. The idea of Partial Products is to start with the largest place value and move to the smallest (the reverse of many standard algorithms). Here are examples for 27×4 :

Partial Products by Place Value

$$27 \times 4 =$$

$$20 \times 4 = 80$$

$$7 \times 4 = 28$$

$$80 + 28 = 108$$

Break Apart Into Addends

$$27 \times 4 =$$

$$25 \times 4 = 100$$

$$2 \times 4 = 8$$

$$100 + 8 = 108$$

Partial Products can be used with large factors, and both factors can be broken into partials. In such cases, rectangles are often used to keep track of each partial product.

$$347 \times 7$$

	300	40	7
7	2,100	280	49

$$\begin{array}{r}
 2,100 \\
 280 \\
 + 49 \\
 \hline
 2,429
 \end{array}$$

$$68 \times 36$$

	60	8
30	1,800	240
6	360	48

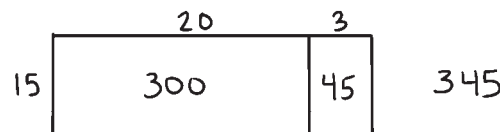
$$\begin{array}{r}
 1,800 \\
 360 \\
 240 \\
 + 48 \\
 \hline
 2,448
 \end{array}$$

Note that the rectangles only roughly represent the size of the numbers; they are not meant to be proportional representations.

HOW DOES PARTIAL PRODUCTS WORK?

Partial Products utilizes the distributive property, breaking apart one or both factors into addends, multiplying the parts, and putting them back together. For 347×7 , this is what happens numerically: $7(300 + 40 + 7) = 2,100 + 280 + 49$. When there are 2 two-digit numbers, one or both can be broken apart. The decision relies on whether the two-digit number can be computed mentally as-is. For example, for 68×36 , neither 68×3 or 6×36 is likely to be recalled mentally. So then, both factors are decomposed.

Now try 23×15 .



There are products you likely recognize: $20 \times 15 = 300$ and $3 \times 15 = 45$. Because of this, you might only decompose one factor (23).

WHEN DO YOU CHOOSE PARTIAL PRODUCTS?

Partial Products is very versatile and always works. However, it can require more steps and take more time than other strategies like Halve and Double or Compensation depending on the factors. For example, 99×5 can be solved with Partial Products ($90 \times 5 + 9 \times 5$), but Compensation is likely a bit easier ($100 \times 5 - 5$). Partial Products would be selected after other strategies are ruled out.

The standard algorithm is also an application of partial products, but the standard algorithm begins with the ones place and works out to larger place values and combines partial products per line. The Partial Products strategy is as versatile as the standard algorithm and oftentimes more accessible to students (because they can use the area model), so it is almost always a good choice for more complicated multiplication. That said, Partial Products can become unwieldy as the number of digits in each factor increases because there are more and more partials to keep track of.

MODULE
1

Break Apart to
Multiply
Strategy

MODULE
2

Halve and
Double
Strategy

MODULE
3

Compensation
Strategy

MODULE
4

Partial
Products
Strategy

MODULE
5

Think
Multiplication
Strategy

MODULE
6

Partial
Quotients
Strategy

MODULE
7

Standard Algorithms
for Multiplication
and Division

PARTIAL PRODUCTS: Strategy Brief for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. This strategy brief is a tool for doing that. You can include it in parent or school newsletters or share it at parent conferences. It is available for download so that you can adjust it as needed.

Partial Products

How It Works: We can break apart factors by place value to multiply. The parts of each factor are multiplied by each of the parts of the other factor. The partial products are added together to get your answer.

1. Decompose each factor.
2. Multiply the parts of each factor by each of the parts of the other factor.
3. Add each of the partial products together to get your final answer.

In the example, 26×47 , each factor is decomposed by place value: $20 + 6$ and $40 + 7$. Then, each part is multiplied by each part of the other factor. 20 is multiplied by 40. 20 is also multiplied by 7, since those are the parts from the other factor. Also, 6 is multiplied by 40 and 6 is multiplied by 7, since those are the parts from the other factor. The partial products are then added together.

The example on the right is an area model. The rectangle represents how the area of these two factors can be decomposed.

When It's Useful: Partial Products is a useful strategy for two- and three-digit factors. It is not as useful when the numbers get larger.

Factor	Factor
26	47
/	\
(20+6)	(40+7)

$20 \times 40 = 800$	Partial Products
$6 \times 40 = 240$	
$20 \times 7 = 140$	
$6 \times 7 = 42$	

↓

product = 1,222

Factor	Factor
26	47

	40	7	
20	800	140	Partial Products
+			
6	240	42	

800
240
140
+ 42
1,222



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

TEACHING ACTIVITIES for Partial Products

Partial Products is the same thing as the Break Apart (with addition) to Multiply strategy. In this module, we provide activities that focus solely on breaking apart by place value as an extension of the break apart activities in Module 1. This strategy bridges initial decomposition strategy with standard algorithms in the future. In this section, we focus on instructional activities for developing the Partial Products strategy. The goal is for students to become adept with the strategy and also to consider when the strategy is useful. Because of its relationship to the Break Apart strategy, many of the activities in Module 1 can be modified for use with the Partial Products strategy.



ACTIVITY 4.1 MULTIPLES MATRIX

Both estimation and computation of multidigit whole numbers rely on the foundation of students' capability to multiply and divide by 10, 100, 1,000, and their multiples (e.g., 40, 400, 4,000). However, this capability is not the rote recall of moving zeros or place values. Rather, *such a* shortcut is grounded in understanding and explicit connection to and application of basic facts. In this activity, students use basic facts to make examples of related problems. Students are given a basic fact and identify what problems it can help with by thinking of it as ___ groups of ___ tens, ___ groups of ___ hundreds, and ___ groups of ___ thousands.

4 groups of 3	4×3	12
4 groups of 30	4×30	120
4 groups of 300	4×300	1,200
4 groups of 3,000	$4 \times 3,000$	12,000

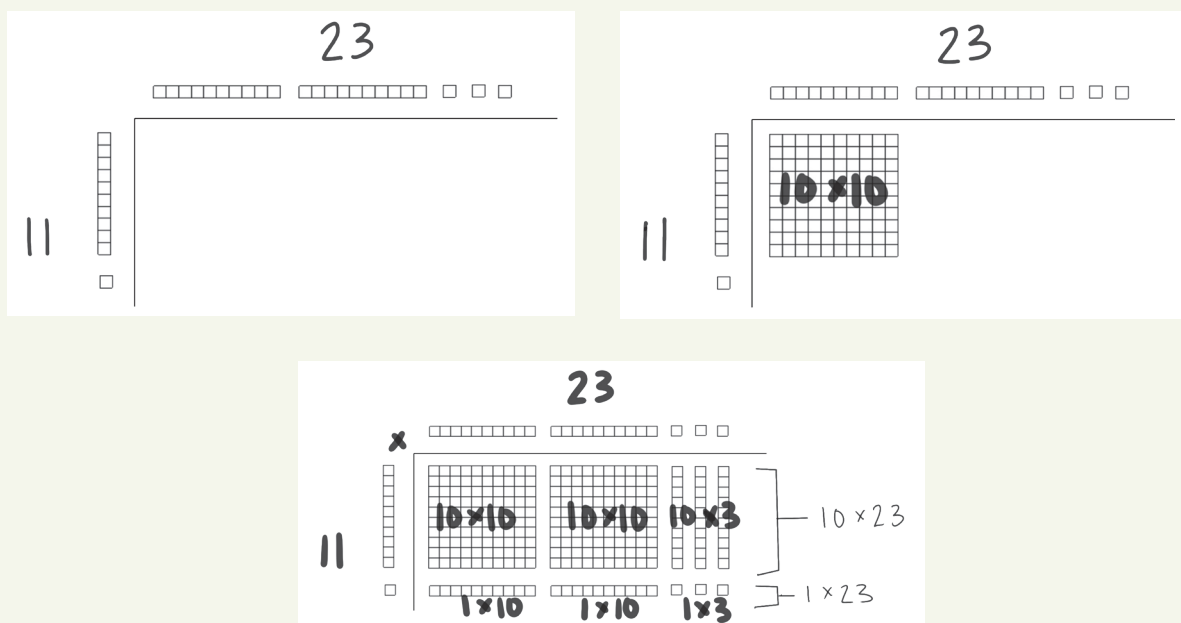
In this example, a group of students worked with the basic fact 4×3 and restated it as groups of tens, hundreds, and thousands before writing the expression and product. Note that it's entirely reasonable to provide students with place value disks or some other tool to find products as they first work with the activity.

Ideally, different groups of students work with three or four different basic facts and record their work on chart paper or something similar. Doing so allows for a gallery walk after all students have completed their work. They can look for similarities, differences, and patterns so they can establish generalizations for working with multiples of 10, 100, and 1,000 as well as how they can leverage basic facts to do so. Note that eventually, students will notice that zeros are "just added" to the product. This is true and useful. You simply want to make sure they understand *why* this is so. As you get into work with Partial Quotients, you can reuse this activity with division facts.



ACTIVITY 4.2 BASE-10 PARTIALS

Base-10 blocks help students make sense of the Partial Products strategy by providing a concrete example. Using manipulatives, properties of operations, and patterns within factors and products can help them develop a deep understanding of multiplication. In this instructional activity, students use base-10 blocks to represent partial products. Start by providing students with one-digit by two-digit expressions. Show the students how to arrange the blocks in an array form. Ensure students have a foundational understanding before moving on to multidigit numbers. The following example models how students could break apart the factors of 23×11 by place value. One side shows 23 split into $10 + 10 + 3$. On the opposite side, students would break apart the 11 into $10 + 1$. Then students would work to fill in the base-10 blocks that would form the array.

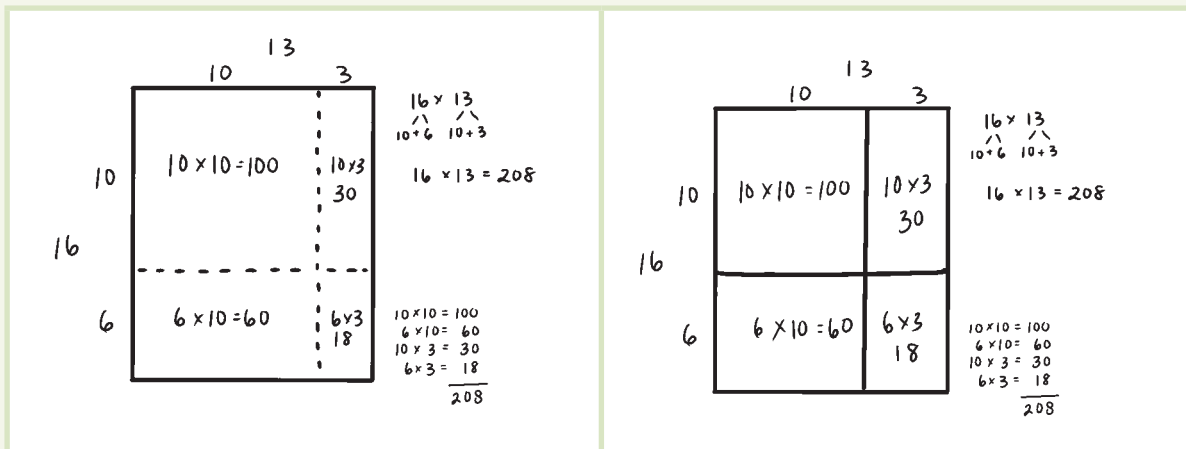


This activity transitions nicely to area models, since it is the physical representation of how students use the area model.



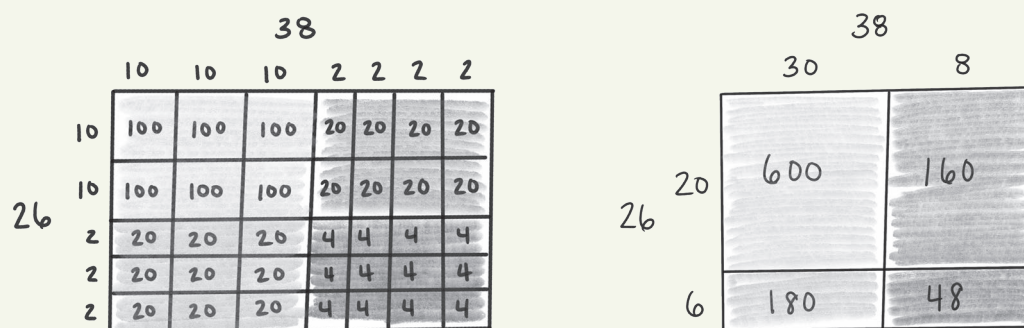
ACTIVITY 4.3 PARTIAL PRODUCTS WITH AREA MODELS

Area models do well to help students make sense of the Partial Products strategy. This is especially true when students have the opportunity to connect precise area models with a more general, open or abstract model. In this activity, students make precise models of multiplication problems using grid paper. Those precise models are connected to open, abstract models and the expressions that represent the partial products. Students will need a few exposures to this activity before shifting to practice with open area models alone. The following example is from a class beginning to work with 2 two-digit factors. The precise model is on the left and the open, general model is on the right. In both examples, the student has shown how the factors are broken apart by place value as well as Partial Products equations.



ACTIVITY 4.4 IS IT THE SAME?

Students may tend to overly rely on breaking apart factors into tens and ones because the multiplication is more comfortable for them. For example, they may decompose 26×3 into $10 \times 3 + 10 \times 3 + 2 \times 3 + 2 \times 3 + 2 \times 3$. They may even decompose the 6 into 6 ones. This can become quite problematic as problems with 2 two-digit factors (26×38) or three-digit by one-digit problems (263×8) become unwieldy. The Partial Products strategy is simply more efficient. This teaching activity focuses on establishing that the two decomposition approaches yield the same result and that the Partial Products strategy is more efficient. In this activity, you pose a problem such as 26×38 . You discuss with students how these two numbers might be broken apart and record their thinking. Then, groups of students work to find the product by decomposing into tens and ones and groups of tens and groups of ones (place value), as shown in the following example. After students find the product, discuss how the models are the same and how they are different. Ask which approach is more efficient or at least easier to keep track of and if they think it will always work. They can then create their own problems to find out.



You can see in the example how students might go about this. Note that this group was permitted to use groups of 2 instead of singles. During discussion, students noted that there are 6 hundreds in the left model and there is a single 600 in the right example but that they are the same. Other students noted that nine groups of 20 is 180 and that there is an area of 180 in the right model. When this was shared, the teacher took time to confirm that nine groups of 20 was, in fact, 180. The same was done for 160 and 48. Students then went off to create new problems to see if this would remain true. This resulted in many different examples for the class to discuss, each showing that when you break apart by place value the product remains the same.



ACTIVITY 4.5 ONE OR BOTH?

The Partial Products strategy allows for one or both factors to be decomposed. Often it makes sense to decompose both. But there are situations in which it is more efficient to decompose only one factor. Those problems include factors that students can easily multiply (like 25) or problems in which no grouping will be needed (22×43). This activity establishes for students that decomposing one factor yields the same results as decomposing both.

Arrange students in groups of two or three. Have half of the groups solve a problem by breaking apart both factors. Have the other groups solve the problem by breaking apart only one of the factors. Be sure that both groups justify their solutions with drawings and equations. Once finished, bring both groups together and have them share their findings. During discussion, ask students questions like these:

- Why do you think the products were the same?
- Does this always work? Why?
- Why would this be useful?
- How did you decide which factor to break apart (for groups that break apart only one)?
- Which approach do you find better? Why?

After discussion, you can pose new problems for students to work with. During the extension, you may choose to have students solve each set of problems with both approaches. Or, you might have groups change the approach they used initially to replicate the initial experience. The activity will work with any set of problems but you might decide to use the following examples:

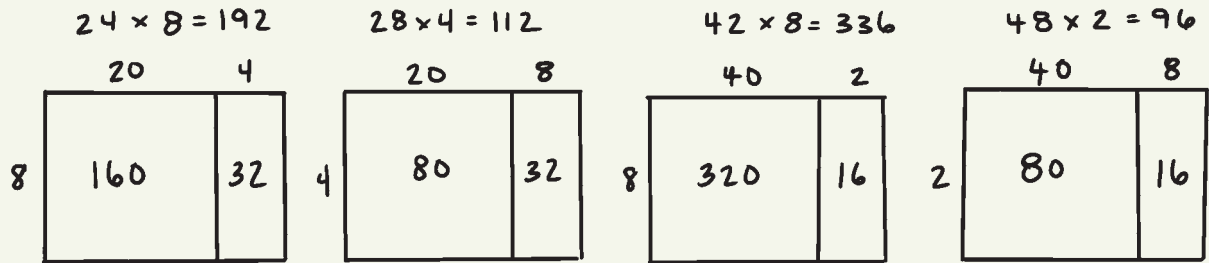
	DECOMPOSING BOTH FACTORS	DECOMPOSING ONE FACTOR
25×43	$(20 \times 40) + (20 \times 3) + (5 \times 40) + (5 \times 3)$	$(25 \times 40) + (25 \times 3)$ OR $(20 \times 43) + (5 \times 43)$
75×52	$(70 \times 50) + (70 \times 2) + (5 \times 50) + (5 \times 2)$	$(75 \times 50) + (75 \times 2)$ OR $(70 \times 52) + (5 \times 52)$
62×31	$(60 \times 30) + (60 \times 1) + (2 \times 30) + (2 \times 1)$	$(62 \times 30) + (62 \times 1)$ OR $(60 \times 31) + (2 \times 31)$
35×55	$(30 \times 50) + (30 \times 5) + (5 \times 50) + (5 \times 5)$	$(35 \times 50) + (35 \times 5)$ OR $(30 \times 55) + (5 \times 55)$



ACTIVITY 4.6 SAME DIGITS, DIFFERENT PRODUCT

Students may misinterpret the commutative property when working with multidigit factors. For example, they may believe that 24×8 is the same as 28×4 . This instructional activity takes aim at this misconception by having students find products and then rearranging digits before finding a new product. Using Partial Products will help them see how and why the products change and that rearranged digits do not yield the same product. For the activity, pose a prompt like this: “ 24×8 , 28×4 , 42×8 , and 48×2 all have the same digits. Do they all have the same product?”

The following example shows how students might go about determining if you can rearrange the digits in factors. Here, the student uses Partial Products and area models to show that the products are not the same.



This activity can be easily modified in a variety of ways. You might ask if 24×8 and 28×4 have the same product in which only the digits in the ones place are swapped. Or you can, like the initial prompt, have students investigate all of the possible combinations for the digits like 24×83 , 24×38 , 23×84 , and 23×48 . Or you could pose one problem like 35×71 and ask students to find different ways to rearrange the digits and determine if any of the products are the same.



ACTIVITY 4.7 PROMPTS FOR TEACHING PARTIAL PRODUCTS

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking, including base-10 models, number lines, arrays, and so on. After students work with the prompt(s), bring the class together to exchange ideas. These could be useful for collecting evidence of student understanding. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once if modified.

- Leticia says that $8 \times 3,240$ has 1,600 as a partial product. Erin says it does not. Who is correct? Explain your answer.
- Why is it important to understand place value when you are multiplying multi-digit numbers?
- Marie used Partial Products to solve 32×73 . She found 30×70 and 2×3 . How do you know if she is finished?
- Connie solves 31×7 by thinking of $10 \times 7 + 10 \times 7 + 10 \times 7 + 1 \times 7$. How else might she have approached the problem?
- Is the product of 49×37 over or under 1,200? Explain how you know.
- Tell why 437×8 is larger than 374×8 even though the digits are the same.
- Describe how you use the Partial Products strategy. Use examples to show your thinking.
- Create examples of when the Partial Products strategy isn't efficient. Justify your thinking.
- Sorsha solved 32×45 by thinking of $30 \times 45 + 2 \times 45$. Deryn solved it by thinking $30 \times 40 + 2 \times 40 + 30 \times 5 + 2 \times 5$. Who do you agree with? Explain your thinking.
- Kaz and O'Neil disagree about the best way to solve 66×42 . Kaz thinks $60 \times 42 + 6 \times 42$. O'Neil thinks $60 \times 40 + 60 \times 2 + 6 \times 40 + 6 \times 2$. Who do you agree with? Explain your thinking.
- Create a story problem for 42×13 . Solve the problem using Partial Products.

NOTES

PRACTICE ACTIVITIES for Partial Products

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the Partial Products strategy or are they reverting to previously learned and/or possibly less appropriate strategies? (e.g., Do they break apart a factor of 54 into 50 and 4 or 10, 10, 10, 10, 10, and 4?) - Are students using the Partial Products strategy efficiently? - Do they decompose one or both factors as appropriate? - Do they use Partial Products regardless of its appropriateness for the problem at hand?
Flexibility	- Are students carrying out the strategy in flexible ways? (e.g., Are they more comfortable breaking apart two-digit factors rather than three-digit factors?) - Do they change their approach to or from this strategy as it proves inappropriate or overly complicated for the problem?
Accuracy	- Are students using the Partial Products strategy accurately (e.g., decomposing factors accurately, finding all partial products)? - Are students finding accurate solutions (e.g., adding partials correctly)? - Are they considering the reasonableness of their solutions?* - Are students estimating before finding solutions?*

* This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

Like Break Apart to Multiply, here are some common challenges for Partial Products:

- The student breaks apart as they would for addition, multiplying tens by tens and ones by ones, but omits the other parts.
 - 54×67 : multiplies $50 \times 60 = 3,000$ and $4 \times 7 = 28$, then adds them to get 3,028.
- The student breaks apart a factor but not into parts that can be readily multiplied mentally.
 - 25×48 : breaks this apart into $20 \times 48 + 5 \times 48$ and gets stuck.
- The student becomes adept at breaking apart by all the parts and doesn't notice when a more efficient option fits the problem.
 - 350×15 : breaks this apart into $(300 + 50)(10 + 5) = 3,000 + 1,500 + 500 + 250$, rather than breaking apart just one factor, such as $350(10 + 5)$, to solve.

4. The student has place value challenges when breaking numbers apart.

- 804×5 : multiplies 80×5 (400) and adds it to 4×5 (20) to get 420.

Partial products are often determined by place value but there are other options, which can be highlighted through worked examples. Comparing both options can help students gain insights into how to use the strategy. The prompts from Activity 4.7 can be used for collecting worked examples. Throughout the module are various worked examples that you can use as fictional worked examples. A sampling of additional ideas is provided in the following table.

SAMPLE WORKED EXAMPLES FOR PARTIAL PRODUCTS		
Correctly Worked Example (make sense of the strategy) What did _____ do? Why does it work? Is this a good method for this problem?	Angelica's work for 240×12 : 240×12 $200 \times 10 = 2,000$ $40 \times 10 = 400$ $200 \times 2 = 400$ $40 \times 2 = 80$ $\underline{2,880}$	Theresa's work for 240×12 : 240×12 $240 \times 10 = 2,400$ $240 \times 2 = 480$ $\underline{2,880}$
Partially Worked Example (implement the strategy accurately) Why did _____ start the problem this way? What does _____ need to do to finish the problem?	Paisley's start for $1,135 \times 4$: $1,135 \times 4$ $1,100 \times 4 = 4,400$	Leo's start for $1,525 \times 42$: $1,525 \times 42$ $1,525 \times 2 = 3,050$ $1,525 \times 4 = 6,100$
Incorrectly Worked Example (highlight common errors) What did _____ do? What mistake does _____ make? How can this mistake be fixed?	Karen's work for 78×45 : 78×45 $70 \times 40 = 2,800$ $8 \times 5 = 40$ $\underline{2,840}$	Lina's work for 408×90 : 408×90 $400 \times 9 = 3,600$ $8 \times 9 = 72$ $\underline{3,672}$



ACTIVITY 4.8

Name: “Why Not?”

Type: Routine

About the Routine: As students learn new strategies, they may begin to believe that every problem should be solved with the new strategy. Fluency comes about by choosing when a strategy is a good choice for a certain problem. In this routine, students encounter three or four expressions and determine why one or more of them are not good choices for using the Partial Products strategy.

Materials: Prepare three or four problems for students to consider.

- Directions:**
1. Pose three or four expressions for students to think about.
 2. Have students examine each expression and consider if Partial Products would be a good strategy for solving the problem. Be sure to ask students to be prepared to justify their thinking.
 3. Then, give students time to discuss their ideas with partners before bringing the whole class together for a group discussion.
 4. Discuss as a whole group. As you discuss student thinking, be sure to recognize that the Partial Products strategy always works. Focus students’ attention on why partials are unnecessary in certain situations.

WHY NOT			
8×31	4×42	5×50	6×39

For example, 25×4 could be thought of as $20 \times 4 + 5 \times 4$ but often we just know that 25×4 is 100. More prevalent examples are those that are clearly related to basic facts such as 70×8 , 60×40 , or 300×9 . In this example, 5×50 is not suited for Partial Products; 6×39 might also be identified because it can be solved with Compensation.

This routine works with any set of problems. These next two sets focus on larger multiplication problems.

WHY NOT			
4×808	295×6	919×7	8×300

WHY NOT			
99×15	25×11	48×12	45×22



ACTIVITY 4.9

Name: *“About or Between”*

Type: *Routine*

About the Routine: Determining reasonableness, through estimation, plays an important role in being fluent. There are different ways to estimate and in this routine, About or Between (SanGiovanni, 2019), students have an opportunity to choose between how they would estimate products.

Materials: Prepare two or three expressions for students to work with.

- Directions:**
1. Pose a problem to students.
 2. Have students determine how they would estimate the product and share their thinking with partners. Students can choose to estimate the products by thinking of nearby friendly numbers (about) or by finding a range (between).
 3. Students share their approaches with partners.
 4. Discuss ideas with the entire class. As you discuss estimates, stress that estimates give an idea of the final result and that there is no one correct way to estimate. Remind students that an efficient method is one that can be done relatively quickly.

THE PRODUCT IS ABOUT OR BETWEEN		
38×5	14×7	46×6

In this example, the class first works with 38×5 . Some students suggest that it is about 200 because 38 is close to 40 and 40×5 is 200. Others say that it is about 150 because 38 is close to 30 and 30×5 is 150, but when they add it will be more than 150. After these are shared, a student suggests that the product of 38×5 will be between 150 and 200 by manipulating the first factor.

THE PRODUCT IS ABOUT OR BETWEEN	
84×77	15×37

In another example, a teacher only poses two expressions because of the size of the factors. For the first expression, students offer that it will be between 80×70 and 90×80 . Other students say it will be about 80×70 . In each of these cases, students offer why each option is useful for estimating the product.



ACTIVITY 4.10

Name: *Make It Close*

Type: *Game*

About the Game: *Make It Close* is a target-based game for practicing multiplication and developing both number and operational sense. The unique twist in this game is that the target changes from round to round. The player closest to the target wins the round.

Materials: four decks of digit cards or playing cards (aces equal to 1, face cards and 10s removed); *Make It Close* recording sheet

TEACHING TAKEAWAY

When more than two players play a game, change features of the game to minimize disengagement.

- Directions:**
1. Players deal four cards to make 2 two-digit factors. Both players use Partial Products to find the product, which becomes the target for the round. In the example, the target is 2,562.
 2. Each player deals themselves four digit cards to make a multiplication problem with 2 two-digit factors.
 3. The players arrange the four digits so that their product is close to the target. Player 1 arranges digits for a product of 2,698. Player 2 arranges digits for a product of 2,058.
 4. The player who has a product closest to the target for round 1 gets a point. This is player 1 in the example.
 5. The first player to 5 points wins the game.

Target	4	2	x	6	1	2,562
--------	---	---	---	---	---	-------

Player 1	3	8	x	7	1	2,698
----------	---	---	---	---	---	-------

Player 2	4	9	x	4	2	2,058
----------	---	---	---	---	---	-------

RESOURCE(S) FOR THIS ACTIVITY

Make It Close	
Directions: Use digit cards to make a multiplication problem. The product of the problem is the target. Deal new digit cards to make a multiplication problem that is close to the target. The player closest to the target gets a point.	
Target Problem	My Problem (give yourself a V if you are closest to the target)

1	2	3	4
5	6	7	8
9	1	2	3
4	5	6	7
8	9	0	0



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.11

Name: *Connect Four Partial*s

Type: *Game*

About the Game: *Connect Four Partial*s is an opportunity for students to apply problem-solving to Partial Products as they generate factors in order to find the partial products.

Materials: *Connect Four Partial*s game board and recording sheet, a 10-sided die, and two-color counters

- Directions:**
1. Player 1 places counters on the game board, making a four-digit number in expanded form.
 2. Player 1 rolls a number.
 3. Player 1 then multiplies the partials they chose on the game board by the number rolled.
 4. Player 1 writes the expression they created and shows their work on the recording sheet.
 5. Player 2 checks their partner's work and then uses a calculator to confirm. If player 1's answer is incorrect, player 1 removes all of their chips from the board. If correct, the counters stay on the board and it is player 2's turn.
 6. Player 2 repeats the steps above.
 7. The game ends when all spaces are covered. The winner is the player with the most rows of four.

For example, player 1 placed counters on 3,000, 400, 50, and 8. They rolled a 5 and wrote the expression $3,458 \times 5$ and solved it on the recording sheet. Player 2 placed their counters on 6,000, 500, 40, and 1 and rolled a 3. Once all spaces are filled, players count the number of rows of four they have.

RESOURCE(S) FOR THIS ACTIVITY

Connect Four Partials

1,000	6,000	100	600	10	60	1	6
2,000	7,000	200	700	20	70	2	7
3,000	8,000	300	800	30	80	3	8
4,000	9,000	400	900	40	90	4	9
5,000	0	500	0	50	0	5	0

Connect Four Partials

Counters Placed	Number Rolled	Partial Products	Final Product
3,000 400 50 8	5	$3,000 \times 5 = 15,000$ $400 \times 5 = 2,000$ $50 \times 5 = 250$ $8 \times 5 = 40$	17,290



Game boards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.12

Name: *Target 2,500*

Type: *Game*

About the Game: This target-based game is an opportunity to estimate and find products of two-digit numbers. *Target 2,500* also offers the opportunity to observe how changing factors influence products. The target of 2,500 is set because 50×50 is half of 100×100 . Players get points for getting close to the target each round.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed) and *Target 2,500* recording sheet

TEACHING TIP

Many recording sheets are optional. You can have students use a piece of paper in their Math Journals to keep track of play.

- Directions:**
1. Each player pulls four digit cards to make 2 two-digit factors.
 2. Each player arranges the digit cards to create a product as close to 2,500 as possible.
 3. The player closest to 2,500 gets a point.
 4. Then, each player can rearrange their digit cards to get closer to 2,500.
 5. The player closest to 2,500 after rearranging earns another point. Note that each player must rearrange regardless of how close they are on the first try.
 6. The first player to 10 points wins the game.

RESOURCE(S) FOR THIS ACTIVITY

Target 2,500

Directions: Use four digit cards to make a multiplication problem with a product as close to 2,500 as possible. Then, rearrange the digits to make another problem with a product as close to 2,500 as possible.

First Problem	Second Problem (after rearranging digits)



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.13

Name: *What's the Problem?*

Type: *Center*

About the Center: This center is an opportunity for students to apply problem-solving as they generate parts of a factor in order to find the partial products and eventually solve the original problem.

Materials: tens cards, digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), What's the Problem? recording sheet

- Directions:**
1. Students pull tens cards to generate a number for the ?? spaces on the area model.
 2. Students pull a digit card to generate a number for the ? on the area model.
 3. Students find the partial products and record the original multiplication problem.

RESOURCE(S) FOR THIS ACTIVITY

What's the Problem?

Directions: Pull a tens card and use it for the double question marks (??). Pull a digit card for the question mark (?). Find the partial products and then write what the original problem was.

	30	?
??		
7	210	

Original problem:

	??	?
??		
4		20

Original problem:

10	20	30	40
50	60	70	80
90	10	20	30
40	50	60	70
80	90	0	0



Tens cards, digit cards, and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.14

Name: **Pick Your Partial**

Type: **Center**

About the Center: This center helps students consider different partial products to solve a multiplication problem.

Materials: Pick Your Partial recording sheet

- Directions:**
1. Provide students with the Pick Your Partial recording sheet.
 2. Students circle the partial products used to solve each expression.

For example, a student has the expression 3×625 . The student has several partials to choose from. The student multiplies 3×600 , which equals 1,800. The student would circle the 1,800 on their recording sheet. Then the student multiplies 3×20 , which equals 60. The student would circle the 60 on their recording sheet. Lastly, the student multiplies 3×5 , which equals 15. The student would circle the 15 on their recording sheet. The student finished the problem. However, there were other choices the student could have made. The student could have picked 3×600 and 3×25 . This would be fewer partials but would still result in the same product.

3 × 625											
180	1800	6	60	150	15	75	600	25	300		
$\begin{array}{r} 625 \\ \times 3 \\ \hline 1800 \\ 60 \\ + 15 \\ \hline 1875 \end{array}$ (3×600) (3×20) (3×5) $\begin{array}{r} 625 \\ \times 3 \\ \hline 1800 \\ + 75 \\ \hline 1875 \end{array}$ (3×600) (3×25)											

RESOURCE(S) FOR THIS ACTIVITY

Pick Your Partial											
Directions: Circle the partial products to solve each expression.											
3 × 625											
180	1,800	6	60	150	15	75	600	25	300		
5 × 727											
700	3,500	7	350	70	30	35	100	27	300		
9 × 327											
288	6	63	180	2,700	81	1	630	3	270		
6 × 1,842											
6,000	10	4,800	48	240	60	12	600	480	24		
3 × 4,261											
1,200	12,000	18	600	30	80	180	120	12	3		
5 × 4,265											
20,100	2,000	25	300	250	200	1,000	30	20,000	10		



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.15

Name: *Guess and Check—The Largest Product*

Type: *Center*

About the Center: Estimating products is a way to help determine the reasonableness of a solution. In this center, students estimate and compare results and then check their thinking using partial products.

Materials: two sets of digit cards (0–9), playing cards (queens = 0, aces = 1, kings and jacks removed), or 10-sided dice; Guess and Check—The Largest Product recording sheet

- Directions:**
1. Students select digit cards to make multiplication expressions. Students pull three cards to make two-digit by one-digit problems, four cards for two-digit by two-digit problems, and so on.
 2. Students estimate which expression will have the largest product and which will have the smallest product.
 3. Students use Partial Products to find the actual products and to see if their estimates were correct.

For example, Stephon creates these problems: 37×84 , 15×46 , and 33×23 . He thinks 37×84 will have the greatest product and that 15×46 will have the smallest product.

RESOURCE(S) FOR THIS ACTIVITY

Guess and Check—The Largest Product			
Directions: Use digit cards to make three multiplication problems. Decide which problem will have the largest product and which will have the smallest product. Use Partial Products to find the exact products and to see if your guesses were correct.			
Problem	Largest Product or Smallest Product	How I Found the Exact Product With Partial Products	Was My Guess Correct?



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.16

Name: *Bag of Blocks*

Type: *Center*

About the Center: Base-10 blocks can be a good tool for helping students see equal groups. This center presents students with a bag of base-10 blocks that shows one group of a certain amount. They then find how many would be in ___ bags (groups) of that amount.

Materials: bags of base-10 blocks, Bag of Blocks recording sheet, 10-sided die

- Directions:**
1. Students empty a bag and count the amount of blocks in the bag.
 2. They record the amount.
 3. Students then roll a number to find how many blocks would be in (the number rolled) bags.
 4. Students use the Partial Products strategy and record their thinking.

TEACHING TAKEAWAY

Many centers can be used first as teaching activities. This provides an additional activity and it helps ensure students know how the center works.

RESOURCE(S) FOR THIS ACTIVITY

Bag	Number of Blocks in the Bag	Number of Bags	Product (showing Partial Products)
A	234	4	$200 \times 4 = 800$ $30 \times 4 = 120$ $4 \times 4 = 16$ $800 + 120 + 16 = 936$



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 4.17

Name: *One or Both?*

Type: *Center*

About the Center: We break numbers apart by their place value to use the Partial Products strategy. One or both factors can be decomposed. Occasionally, students overly generalize this strategy and assume that each factor has to be broken apart. This center helps students see that either or both factors can be decomposed. One or Both is a follow-up center for Activity 4.5 (“One or Both?”).

Materials: One or Both multiplication expression cards and recording sheet

- Directions:**
1. Students select a multiplication expression card.
 2. Students find the product by breaking apart both factors and show their thinking with partial products.
 3. Students find the product of the same problem by breaking apart one of the factors and show their thinking with partial products.
 4. Students determine which approach was easiest for them to complete and circle that solution.

Possible expression cards are available to download.

RESOURCE(S) FOR THIS ACTIVITY

One or Both		
Directions: Select a problem. Solve it by breaking apart both factors. Then, solve it by breaking apart one factor. Use drawings and numbers to show your thinking.		
Problem	Break Apart Both Factors	Break Apart One Factor

32×46	22×84
19×46	32×63
25×66	23×65



Expression cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

[illegible]

Think Multiplication Strategy

STRATEGY OVERVIEW:

Think Multiplication

What is Think Multiplication? This is a division strategy wherein, as the title suggests, the problem is reframed as a multiplication problem with a missing factor. For example, $240 \div 3 = ?$ is thought of as $? \times 3 = 240$. Think Multiplication is first introduced as a strategy for recalling basic division facts. For $24 \div 3$, a student would ask themselves, “What times 3 equals 24?” Knowing that the missing factor is 8, they then translate the missing factor to be the quotient of the original problem. The same thinking applies as numbers get larger, as illustrated in the following examples.

DIVISION PROBLEM	STUDENT THINKING	THINK MULTIPLICATION PROBLEM
$24 \div 3 = ?$	How many groups of 3 in 24? or What times 3 equals 24?	$? \times 3 = 24$
$249 \div 3 = ?$	How many groups of 3 in 249?	$? \times 3 = 249$
$475 \div 25 = ?$	How many 25s in 475?	$? \times 25 = 475$

HOW DOES THINK MULTIPLICATION WORK?

Think Multiplication is an example of using an inverse relationship. Symbolically, this means that if $a \times b = c$, then $c \div b = a$ and $c \div a = b$. For $249 \div 3 = ?$, it can be interpreted as “How many groups of 3 in 249?” or “How many rows of 3 to have 249?” If the missing factor is known, then that is the end. If it is not known, then Partial Quotients can be employed. Here are some examples:

$78 \div 6 \rightarrow$ How many 6s in 78? I don't know, so I will break apart the dividend into $60 + 18$. How many 6s in 60? 10 How many 6s in 18? 3 The answer is 13.	$330 \div 15 \rightarrow$ How many 15s in 330? I don't know. I can break apart 330 into $150 + 150 + 30$. How many 15s in 150? 10 How many 15s in 30? 2 The answer is 22.
--	--

WHEN DO YOU CHOOSE THINK MULTIPLICATION?

Anytime there is a division situation, Think Multiplication is one of only three options. It is a particularly good option when there is an overt relationship to a basic fact. The following examples of division problems clearly relate to $6 \times 6 = 36$ and can be found using that known fact.

$36 \div 6$	$360 \div 6$	$360 \div 60$	$3,600 \div 6$	$3,600 \div 60$	$3,600 \div 600$
-------------	--------------	---------------	----------------	-----------------	------------------

Another good opportunity for Think Multiplication is when the divisor is a well-known multiple or automaticity (i.e., 15s, 25s, 30s), such as $45 \div 15$ or $375 \div 25$, or is another divisor for which the multiples are well known, like 250 ($1,800 \div 250$).

THINK MULTIPLICATION: Strategy Briefs for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. These strategy briefs are a tool for doing that. You can include them in parent or school newsletters or share them at parent conferences. They are available for download so that you can adjust them as needed.

Think Multiplication

How It Works: We can use the relationship between multiplication and division to solve division problems. Division is also a missing factor multiplication problem.

1. Rethink the division problem as a missing factor multiplication problem.
2. Use multiplication to find the unknown factor.
3. Once known, the factor becomes the quotient of the original problem.

The left example shows that the problem is thought as $6 \times ? = 366$. The relationships between 366 and 6 are recognized. $6 \times 60 = 360$. $6 \times 1 = 6$. $6 \times 61 = 366$, so $366 \div 6 = 61$.

The right example is similar. 154 is recognized as 140 and 14, which are both products of 7.

When It's Useful: Think Multiplication is useful when division problems have clear relationships to multiplication problems.

Dividend	Divisor	
366	6	$= ?$
Factor	$6 \times \boxed{?} = 366$	
	$6 \times \boxed{60} = 360$	
	$6 \times \boxed{1} = 6$	
	$366 \div 6 = 61$	Quotient

Dividend	Divisor	
154	7	$= ?$
Factor	$7 \times \boxed{?} = 154$	
	$7 \times \boxed{20} = 140$	
	$7 \times \boxed{2} = 14$	
	$154 \div 7 = 22$	Quotient

Think Multiplication

How It Works: We can use the relationship between multiplication and division to solve division problems. Division is also a missing factor multiplication problem.

1. Rethink the division problem as a missing factor multiplication problem.
2. Use multiplication to find the unknown factor.
3. Once known, the factor becomes the quotient of the original problem.

The left example shows that the problem is thought as $30 \times ? = 240$. The person then uses a known basic fact (3×8) to find the related problem (30×8). The unknown factor, 30, is the quotient of the original problem.

The right example does not rely on a basic fact but a familiar problem of 12×2 . Several equations were used to find the missing factor.

When It's Useful: Think Multiplication is useful when division problems have clear relationships to multiplication problems.

Dividend	Divisor	
240	30	$= ?$
Factor	$30 \times \boxed{?} = 240$	
	$3 \times 8 = 24$	
	$30 \times \boxed{8} = 240$	
	$240 \div 30 = 8$	Quotient

Dividend	Divisor	
240	12	$= ?$
Factor	$12 \times \boxed{?} = 240$	
	$12 \times 2 = 24$	
	$12 \times \boxed{20} = 240$	
	$240 \div 12 = 20$	Quotient



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

NOTES

MODULE

1

Break Apart to
Multiply
Strategy

MODULE

2

Halve and
Double
Strategy

MODULE

3

Compensation
Strategy

MODULE

4

Partial
Products
Strategy

MODULE

5

Think
Multiplication
Strategy

MODULE

6

Partial
Quotients
Strategy

MODULE

7

Standard Algorithms
for Multiplication
and Division

TEACHING ACTIVITIES for Think Multiplication

Before students are able to choose strategies, a key to fluency, they first must be able to understand and use relevant strategies. These activities focus specifically on Think Multiplication. Think Multiplication relies on understanding how multiplication and division are related and recognizing basic facts within dividends and divisors. This strategy builds toward standard algorithms in the future. While students may employ other methods, which is appropriate, they also must learn this important strategy to support their flexibility and efficiency. In this section, we focus on instructional activities for developing understanding of Think Multiplication and consideration for when to use it.

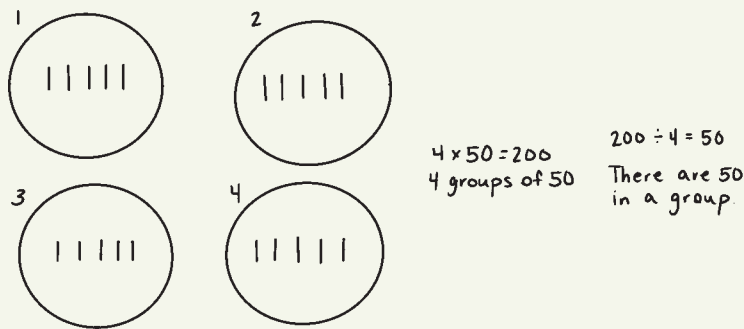


ACTIVITY 5.1 IS IT STILL TRUE?

Students are introduced to the inverse relationship between multiplication and division when they first learn about the operations and as they learn strategies for recalling basic facts. Although students may show solid understanding of this concept with basic facts, they may be challenged to transfer it to multidigit situations. This activity is an opportunity for students to revisit this inverse relationship to prove that it is still true. Here is an example of the prompts you might post.

IT IS TRUE THAT ...	BUT IS IT TRUE THAT ... ?
$5 \times 4 = 20$ is the inverse of $20 \div 4 = 5$	$50 \times 4 = 200$ is the inverse of $200 \div 4 = 50$
$5 \times 4 = 20$ is the inverse of $20 \div 4 = 5$	$5 \times 40 = 200$ is the inverse of $200 \div 40 = 5$
$8 \times 3 = 24$ is the inverse of $24 \div 3 = 8$	$800 \times 3 = 2,400$ is the inverse of $2,400 \div 3 = 800$
$8 \times 3 = 24$ is the inverse of $24 \div 3 = 8$	$8 \times 300 = 2,400$ is the inverse of $2,400 \div 300 = 8$

For these examples, students work collaboratively to determine if the related statement is true. However, at this time in their learning, they cannot focus on the number of zeros in the dividends, divisors, and quotients. Instead, they should use place value models or drawings to prove their thinking. Students might draw 5 tens in four groups along with the multiplication and division equations that their drawing represents.



After students work to prove the relationships, you should lead a discussion with the group about how the equations in the left column are related to those in the right and how that can be useful when multiplying and dividing.



ACTIVITY 5.2

THE DIRECT CONNECT: THREE STRINGS

This activity is another experience for students to make connections between basic facts, multiplication, and division. It helps them see patterns as you eliminate everything but the related problems as you would in a number string. The twist with this activity is that it presents three number strings that are connected through the inverse relationship of multiplication and division. As this is an early experience, do not expect students to quickly solve the problems. Instead, ask groups to solve each problem within the table using tools, drawings, and numbers or equations. Then, convene the groups to share their findings for each equation. During discussion, first confirm the missing values. But then, focus students' attention on the relationship between the multiplication and division problems.

$3 \times 8 =$	$3 \times 80 =$	$3 \times 800 =$	$3 \times 8,000 =$	$30 \times 80 =$	$30 \times 800 =$
$24 \div 3 =$	$240 \div 3 =$	$2,400 \div 3 =$	$24,000 \div 3 =$	$2,400 \div 30 =$	$24,000 \div 30 =$
$24 \div ? = 3$	$240 \div ? = 3$	$2,400 \div ? = 3$	$24,000 \div ? = 3$	$2,400 \div ? = 30$	$24,000 \div ? = 30$

In this example, you might ask debriefing questions like these:

- How did knowing 3×8 help you solve all of the problems, including the division problems?
- What patterns did you notice in the problems?
- Why does it make sense that $3 \times 8 = 24$ and so 3×8 tens equals 24 tens or 240? Where do you see this thinking in other problems?
- Tell how you used any multiplication problem to help you solve the division problem in that column.
- Which of the problems were most challenging to think about? Why?
- Now that you have done this, what multiplication and division problems could you make if I gave you the basic fact $6 \times 3 = 18$?

The last bulleted question is an example of how you can extend the activity for other lessons. You can also modify it a bit to become a more problem-based variation that students would work with after they have worked with the first version a few times.

$7 \times 5 =$	$? \times ? = 350$	$? \times ? = 3,500$	$? \times ? = 3,500$	$? \times ? = 35,000$	$? \times ? = 35,000$
$35 \div 7 = ?$	$350 \div 7 = ?$	$3,500 \div 7 = ?$	$3,500 \div 70 = ?$	$35,000 \div 7 = ?$	$35,000 \div 70 = ?$
$35 \div ? = 7$	$350 \div ? = 7$	$3,500 \div ? = 7$	$3,500 \div ? = 70$	$35,000 \div ? = 7$	$35,000 \div ? = 70$

This table shows what that variation might look like. Notice that the top row has the multiplication fact but the other problems have unknown factors. Students use the information in the related division facts to determine the unknowns in each column.



ACTIVITY 5.3 RECOGNIZING PRODUCTS AND DECOMPOSING DIVIDENDS

Using the inverse is helpful in division situations beyond known multiplication problems. In some problems, such as $154 \div 7$, you can decompose the dividend into known multiples of the divisor. For example, 14 is a familiar multiple of 7; 154 can be decomposed into 140 (10 groups of 14) and 14. Then, it can be reasoned that $154 \div 7$ is 20 groups of 7 and two more groups. This use of Think Multiplication is very handy but it takes time to develop. It will need to be taught and practiced. You may have to be more direct when teaching it by giving a direct prompt for students to explore. For example, you might pose this example:

Kristen thinks she can find $72 \div 6$ by thinking of $60 \div 6 + 12 \div 6$. Prove why Kristen is correct or incorrect. Create other examples to show how her thinking will or won't work.

As you might imagine, students will embark on proving this in a variety of ways. Look for students who rely simply on expressions and equations to prove their thinking as in the example. This may signal those who are advancing to a point that they can make arguments symbolically.

$$\begin{array}{r}
 72 \div 6 \\
 60 \div 6 = 10 \\
 12 \div 6 = 2 \\
 \hline
 72 \quad \quad 12
 \end{array}
 \qquad
 \begin{array}{l}
 \text{So Kristen} \\
 \text{is right.} \\
 72 \div 6 = 12
 \end{array}$$

Also look for students who stray from the original prompt and attempt to decompose the divisor. As you know, the divisor cannot be decomposed. This must be explored and discussed if it comes up in student work. If it does not, you will need to provoke it so that students are aware that it doesn't work. You can use a prompt like this:

Brian thinks he can find $72 \div 6$ by thinking $72 \div 2 + 72 \div 3$. Prove why Brian is correct or incorrect. Create other examples to show if his thinking will or will not work.

The example proves what you know to be true. This student also uses equations alone to show his thinking, which is fine. In both cases, the student used a calculator to prove the accuracy of his work.

$$\begin{array}{r}
 72 \div 3 = 24 \\
 72 \div 2 = 36 \\
 \hline
 6 \quad 60
 \end{array}
 \qquad
 \begin{array}{l}
 72 \div 6 \text{ does not} \\
 \text{equal } 60
 \end{array}$$

The additional examples that students create in both situations will enrich the discussion and provide further evidence that the dividend can be decomposed. This leads to a very important prompt for you to pose: *Why* is breaking apart the dividend helpful? Students will have all sorts of ideas. Help them "discover" the idea by providing the next few true/false problems to explore, as each helps develop understanding of how to break apart to support division.

$$65 \div 5 \text{ is the same as } 50 \div 5 + 15 \div 5$$

$$48 \div 12 \text{ is the same as } 48 \div 3 + 48 \div 4$$

$$42 \div 3 \text{ is the same as } 30 \div 3 + 12 \div 3$$

$$60 \div 15 \text{ is the same as } 60 \div 5 \text{ and } 60 \div 3$$

$$84 \div 4 \text{ is the same as } 40 \div 4 + 40 \div 4 + 4 \div 4$$

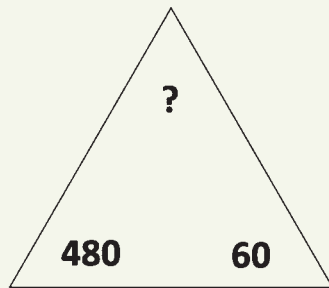
Activity 5.8 ("Breaking Dividends") is a routine for providing more practice with this idea.



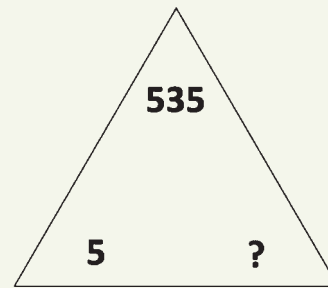
ACTIVITY 5.4 WHAT'S MISSING?

The triangle cards used in this activity vary slightly from triangle fact cards that students are familiar with, in that one of the numbers is unknown. In this activity, students determine the missing value on a triangle card using multiplication and division. Students should justify their thinking with pictures and numbers. After finding solutions, students should discuss how they found their solutions and should share the resulting multiplication and division equations that the cards represent. During this discussion, reinforce that think multiplication can be an effective strategy for finding quotients.

THINK MULTIPLICATION TRIANGLE CARDS FEATURING BASIC FACT CONNECTIONS



THINK MULTIPLICATION TRIANGLE CARDS WITH DIVIDENDS THAT CAN BE DECOMPOSED INTO MULTIPLES OF THE DIVISOR



Note that early work with this strategy makes use of division problems that are clearly connected to basic facts. The card on the left is an example. As students advance, this activity can be used with dividends that are composed of recognizable multiples of the divisor. The card on the right shows this as $535 \div 5$, which can be thought of as $500 \div 5 + 35 \div 5$.



ACTIVITY 5.5 PROMPTS FOR TEACHING THINK MULTIPLICATION

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking, including base-10 models, number lines, arrays, and so on. After students work with the prompt(s), bring the class together to exchange ideas. These could be useful for collecting evidence of student understanding. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once if modified.

- Does the relationship between multiplication and division change when you are no longer working with basic facts? Use examples to justify your thinking.
- Tell how $5 \times 7 = 35$ can help you with $350 \div 5$, $350 \div 7$, $350 \div 50$, and $350 \div 70$.
- Ello solves $105 \div 7$ by thinking of $70 + 35$. How does this help Ello find the quotient?
- Create a story problem for $213 \div 3$.
- How does knowing 4×8 help you solve $3,232 \div 4$?
- Which basic facts does not help you solve $4,218 \div 6$ (6×7 , 6×5 , or 6×3)? Explain your answer.
- Which basic facts help you solve $155 \div 5$ (5×3 , 5×9 , or 5×1)? Explain your answer.
- What is the missing factor in $70 \times \underline{\quad} = 630$ that will help you solve $630 \div 70$?
- How could you break apart 372 to help you solve $372 \div 6$?

PRACTICE ACTIVITIES for Think Multiplication

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the Think Multiplication strategy or are they reverting to previously learned and/or possibly less appropriate strategies like repeated subtraction? - Are students using the Think Multiplication strategy efficiently? (i.e., Do they linger with a dividend trying to determine if it can be decomposed into multiples of the divisor? For example, in $616 \div 7$, 616 is composed of 560 and 56 but that may be a challenging composition for a student to find.) - Do they use the strategy regardless of its appropriateness for the problem at hand?
Flexibility	- Are students carrying out Think Multiplication in flexible ways? (i.e., Do they use it for problems with clear connections to basic facts, such as $350 \div 7$, and problems with reasonable composite dividends, such as $84 \div 7$?) - Do they change their approach to or from this strategy as it proves inappropriate, inaccurate, or overly complicated for the problem?
Accuracy	- Are students using the Think Multiplication strategy accurately? (e.g., They understand $75 \div 25 = ?$ as $? \times 25 = 75$, not $25 \times 75 = ?$) - Are students finding accurate solutions? - Are they recalling and applying basic facts accurately? - Are they considering the reasonableness of their solutions?* - Are students estimating before finding solutions?*

*This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

Worked examples can help students make connections between multiplication and division. The Think Multiplication strategy is commonly used to help students make sense of division. The following, however, are some challenges students may have with the strategy:

- After the student restates the expression as a missing factor equation, they don't just know the answer and are "stuck."
 - $104 \div 8$: interprets this as "How many 8s in 104?" Since that is not a basic fact, the strategy (or problem) is abandoned.

2. The student breaks apart the dividend, but not into parts that are multiples of the divisor:

- $104 \div 8$: breaks this apart into $100 \div 8$ and $4 \div 8$, rather than $80 \div 8$ and $24 \div 8$.

The prompts from Activity 5.5 can be used for collecting worked examples. Throughout the module are various examples that you can use as worked examples. A sampling of additional worked examples is provided in the following table.

SAMPLE WORKED EXAMPLES FOR BREAK APART TO MULTIPLY	
Correctly Worked Example (make sense of the strategy) What did _____ do? Why does it work? Is this a good method for this problem?	Angelica's work: $495 \div 16 \rightarrow$ How many 16's in 495? I don't know products I know $320 \div 16 \rightarrow$ How many 16's in 320? 20 $160 \div 16 \rightarrow$ How many 16's in 160? 10 $320 + 160 = 480$ That leaves 15. Answer : 30 r15
Partially Worked Example (implement the strategy accurately) Why did _____ start the problem this way? What does _____ need to do to finish the problem?	Which way to start: Tensly's way or Daryl's way? $72 \div 6 \rightarrow$ How many 6's in 72? Tensly's Way $(60 \div 6) + (12 \div 6)$ Daryl's Way $(36 \div 6) + (36 \div 6)$
Incorrectly Worked Example (highlight common errors) What did _____ do? What mistake does _____ make? How can this mistake be fixed?	Sarah's work for $3,400 \div 250$: My problem : How many 250's in 3,400? I'll find how many 100's to start: $3,400 \div 100 = 34$ $3,400 \div 100 = 34$ $3,400 \div 50 = 68$ Answer 136



ACTIVITY 5.6

Name: *"If I Know . . ."*

Type: *Routine*

About the Routine: Think Multiplication relies on noticing and using multiplication facts to find quotients. In the "If I Know . . ." routine (SanGiovanni, 2019), students are provided a known multiplication fact that is related to a series of division problems. Students use the known to find all of the quotients. Then, the class talks about how each problem relates back to the known.

Materials: Prepare a series of related problems.

- Directions:**
1. Pose a multiplication fact with the product.
 2. Then, unveil a list of division problems that are related to the multiplication problem.
 3. Have partners work together to determine the unknown in each of the division problems.
 4. After a few moments, bring the class together to discuss what each unknown is and how the known multiplication fact is related to the division problem.

IF I KNOW $3 \times 7 = 21$		
$21 \div 3 =$	$21 \div 7 =$	$21 \div 3 =$
$210 \div 3 =$	$210 \div 7 =$	$42 \div 3 =$
$210 \div 30 =$	$210 \div 70 =$	$84 \div 3 =$
$2,100 \div 3 =$	$2,100 \div 7 =$	$231 \div 3 =$
$2,100 \div 30 =$	$2,100 \div 70 =$	$2,121 \div 3 =$
$2,100 \div 300 =$	$2,100 \div 700 =$	

In this example, the known multiplication fact is 3×7 . The first two columns show division problems that have a clear relationship to the known fact. Typically, the routine would feature these two columns. The third column might be reserved for advanced practice but is offered here to show other possibilities for the routine. The last four dividends connect with 21. The first (42) is twice as much; next is 84, which is twice as much as 42 or 4 times as much as 21; 231 is 210 and 21; while 2,121 is 2,100 and 21 more.



ACTIVITY 5.7

Name: *“The Count Up”*

Type: *Routine*

About the Routine: This routine has students estimate and skip-count. It is a good opportunity for practicing multiples while supporting the Think Multiplication strategy.

Materials: This routine does not require any materials.

- Directions:**
1. Set a clear counting path so that students know how they will count in the room. Having them gather in a large circle is one option.
 2. Pose a division problem for students to consider.
 3. Then, ask students how many counts of the divisor will be needed to get to the dividend. Other questions to ask before counting include the following:
 - What are some numbers we will say as we count?
 - What number do you think you will say?
 - Who will be the last person to count?
 4. Identify the first student to count and then students skip count up by the divisor around the room. As students count, record the numbers they say on the board, as this will help determine how many groups of the divisor are in the dividend and support the debriefing discussion.
 5. Discuss how student predictions about the count compared to the actual count. Then, discuss how the count up connects with the division problem. Be sure to make explicit connections to multiplication.

For example, you pose $85 \div 5 = ?$ Students think about what the quotient might be. Then, they discuss the precount questions you ask, such as the bulleted examples in the third step in the directions above. Then, they count up to 85; while they do so, you record the numbers they say. After the count is complete, you return to the problem and ask students what they thought the quotient was. Pointing at the multiples recorded, count the number of multiples (groups), helping students see that each skip count is one more group. Ask students if they could have thought about counting up differently. In this example, 10 counts of 5 was 50, so there would be 35 more to go; 10 counts of 5 (50) plus 7 counts of 5 (35) equals 17 counts of 5 (85).

In another example, you might pose $420 \div 70 = ?$ Ask students to predict who will be the last person to count and what numbers will be said. During discussion, listen for students who are sharing how the division problem is related to another multiplication problem.



ACTIVITY 5.8

Name: *“Breaking Dividends”* Type: *Routine*

About the Routine: Using the inverse is efficient for problems that are composites of multiples. That is, $72 \div 6$ can be solved by thinking of 72 as a composite of multiples of 6 such as $60 + 12$ or $30 + 42$. Students need lots of practice with this idea. This routine provides an opportunity to do just that. This is an overlap with the Partial Quotients strategy (see Module 6).

Materials: Prepare a few problems for students to consider.

Directions: This routine explicitly focuses on decomposing the dividend to find friendly computations that make use of the inverse relationship.

1. Post or state a divisor to students (e.g., 6).
2. Have students identify multiples of the divisor.
3. Pose the division problem and have partners discuss how they might decompose the dividend into multiples of the divisor to take advantage of the Think Multiplication strategy.

For example, a teacher first asks students to think about multiples of 6 and record their ideas. Then, she poses $72 \div 6$, asking students how they might break apart 72 into multiples of 6. Some may offer that 72 can be thought of as 60 and 12, so they can think of it as $60 \div 6 + 12 \div 6$. Others might say that they see 72 as 36 and 36, so they can think of it as $36 \div 6 + 36 \div 6$. Then, the teacher asks students how they might use their ideas about breaking apart for $84 \div 6$ and then $96 \div 6$. Here are some other examples to use in this routine:

- $84 \div 7$ followed by problems like $91 \div 7$, $98 \div 7$, $105 \div 7$, or $112 \div 7$
- $48 \div 3$ followed by problems like $54 \div 3$, $57 \div 3$, $66 \div 3$, or $72 \div 3$
- $70 \div 5$ followed by $75 \div 5$, $80 \div 5$, $95 \div 5$, $110 \div 5$, or $115 \div 5$
- $90 \div 6$ followed by $96 \div 6$, $102 \div 6$, $126 \div 6$, or $132 \div 6$

Creating expressions for students to work with may seem challenging. Here is one way to generate problems: First, think of a divisor (like 6 in the last example). Then, think of a dividend that is composed of 10 times the divisor (60) and a multiple of the divisor more (36). As students progress, begin to use dividends that use multiples of 10 times the divisor (120).



ACTIVITY 5.9

Name: *The Take*

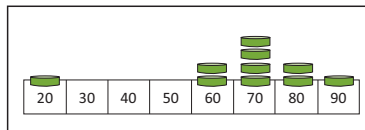
Type: *Game*

About the Game: *The Take* reinforces division problems that have clear connections to multiplication and basic facts.

Materials: *The Take* division cards, 1 set per group; 1 game board per group; and 10 counters per player

- Directions:**
1. Players place 10 counters on their game board. Players can place multiple counters or no counters on any number.
 2. Players take turns choosing a division card.
 3. The player reads the problem and tells the quotient.
 4. The player removes a chip from the space that matches the quotient. The player loses their turn if there is no counter on that space.
 5. The first player to remove all of their counters wins.

For example, a player places their 10 chips on the board as shown. The first card they choose is $320 \div 4$ and they would remove one counter from 80. On their next turn, their card is $350 \div 7$. They don't have a counter on 50, so they lose their turn.



TEACHING TAKEAWAY

To keep both students engaged, have both students work the problem even though it may be their opponent's turn.

RESOURCE(S) FOR THIS ACTIVITY

$160 \div 2$	$480 \div 6$
$240 \div 3$	$560 \div 7$
$320 \div 4$	$640 \div 8$
$400 \div 5$	$720 \div 9$

The Take

Directions: Place 10 counters on the numbered spaces. You can put more than one counter on a number. Pull a division card. Remove a counter if it is on a number that is a quotient of the problem. Be the first to remove all 10 counters.

2	3	4	5	6	7	8	9
---	---	---	---	---	---	---	---

The Take TOO

Directions: Place 10 counters on the numbered spaces. You can put more than one counter on a number. Pull a *The Take TOO* division card. Remove a counter if it is on a number that is a quotient of the problem. Be the first to remove all 10 counters.

20	30	40	50	60	70	80	90
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Game cards and game boards can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.10

Name: *Paper Clip Connections*

Type: *Game*

About the Game: *Paper Clip Connections* is a take on the classic Product Game in which students try to find products by moving their paper clips strategically. The game reinforces the relationship between multiplication and division with larger numbers. As students find products, they must share the related division problem in order to place their game piece.

Materials: *Paper Clip Connections* game board, two paper clips, about 40 two-color counters, and calculators

- Directions:**
1. Player 1 places a paper clip on one of the factors in the left column and one of the factors in the bottom row. Factors are not within the grid.
 2. The player finds the product of these two numbers and then tells their opponent the related division problem.
 3. If accurate, player 1 places their counter on the space of the corresponding product of the two factors.
 4. Player 2 then chooses to move each paper clip one space in either direction and repeats finding the product and telling the related division problem. Or player 2 can move one paper clip two spaces in either direction.
 5. Play continues with each player choosing to move two paper clips one direction or moving one paper clip two spaces in either direction.
 6. The first player to get five counters in a row (horizontally, vertically, or diagonally) wins the game.

For example, player 1 places paper clips on 40 and 30. They say that $40 \times 30 = 1,200$ and so $1,200 \div 40 = 30$. Their opponent confirms and they place their counter on 1,200. Player 2 moves the paper clip on 40 to 30 and the paper clip on 30 to 20. Player 2 says $30 \times 20 = 600$ and so $600 \div 30 = 20$. Player 2 is correct and places their counter on 600. Player 1 then moves the paper clip on 30 two spaces to 50 and cannot move the other paper clip. They share $50 \times 20 = 1,000$ and that $1,000 \div 50 = 20$ and place their counter.

RESOURCE(S) FOR THIS ACTIVITY

Paper Clip Connections

Directions: Player 1 places a paper clip on a number in the column and one outside of the grid. The player picks a counter on the product and makes the related division problem. Player 2 moves one or more paper clips one space in either direction, places their counter on the product, and states the related division problem. The first player to get five in a row wins.

2	40	60	80	100	120	140	160	180
3	60	90	120	150	180	210	240	270
4	80	120	160	200	240	280	320	360
5	100	150	200	250	300	350	400	450
6	120	180	240	300	360	420	480	540
7	140	210	280	350	420	490	560	630
8	160	240	320	400	480	560	640	720
9	180	270	360	450	540	630	720	810
	20	30	40	50	60	70	80	90

Paper Clip Connections

Directions: Player 1 places a paper clip on a number in the column and one outside of the grid. The player picks a counter on the product and makes the related division problem. Player 2 moves one or more paper clips one space in either direction, places their counter on the product, and states the related division problem. The first player to get five in a row wins.

20	400	600	800	1,000	1,200	1,400	1,600	1,800
30	600	900	1,200	1,500	1,800	2,100	2,400	2,700
40	800	1,200	1,600	2,000	2,400	2,800	3,200	3,600
50	1,000	1,500	2,000	2,500	3,000	3,500	4,000	4,500
60	1,200	1,800	2,400	3,000	3,600	4,200	4,800	5,400
70	1,400	2,100	2,800	3,500	4,200	4,900	5,600	6,300
80	1,600	2,400	3,200	4,000	4,800	5,600	6,400	7,200
90	1,800	2,700	3,600	4,500	5,400	6,300	7,200	8,100
	20	30	40	50	60	70	80	90



Game boards can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.11

Name: *Covering the Breaks*

Type: *Game*

About the Game: When using the inverse to divide, it is useful to decompose the dividend into multiples of the divisor. Students need practice with this approach. *Covering the Breaks* is an opportunity for practice with this idea.

Materials: *Covering the Breaks* game board, 10-sided dice, about 30 two-color counters, math journal or scrap paper, and a calculator

- Directions:**
1. Players take turns rolling a digit using a 10-sided die. If the player rolls a 1 or 2, they roll again.
 2. When a player rolls any other number (3–9), the player identifies a number on the game board that can be decomposed into multiples of the number rolled.
 3. The player shares how they would decompose the number and record their thinking.
 4. The player then works the problem to find the quotient of the space selected and the number rolled.
 5. If accurate, the player places their counter on the space and it's their opponents turn.
 6. The first player to get three in a row wins.

For example, player 1 rolls an 8. They choose to cover 96, stating that “96 can be decomposed into 80 and 16.” The player records $80 \div 8 = 10$ and $16 \div 8 = 2$, so $96 \div 8 = 12$ ($10 + 2$). Player 2 then rolls a 4. Player 2 decides to cover 52 because 52 can be thought of as 40 and 12. Player 2 records $40 \div 4 = 10$ and $12 \div 4 = 3$, so $52 \div 4 = 13$. Player 1 then rolls a 4. Player 1 chooses 88, noting that it can be decomposed into 40, 40, and 4, and records their work as described.

TEACHING TIP

To keep students engaged, have both players take their turn at the same time. Here, both players would roll, compute, and record at the same time.

RESOURCE(S) FOR THIS ACTIVITY

Covering the Breaks				
Directions: Take turns rolling a number. Choose a number on the game board to divide by the number you roll. Tell how the number on the space can be broken up to make division efficient. Record your thinking on a piece of paper and show your opponent. If you are correct, place your counter on the space. Get three spaces in a row to win.				
42	52	63	65	66
66	72	75	77	84
84	85	88	91	92
95	96	98	102	104
105	108	110	126	126
132	135	147	154	168



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.12

Name: *Area Model Duel*

Type: *Game*

About the Game: *Area Model Duel* helps students think about ways to efficiently break apart a dividend into convenient numbers. Students show their thinking with an area model.

Materials: *Area Model Duel* cards and expression cards for each group, whiteboards and dry-erase markers for each player

- Directions:**
1. Players place cards face down in two piles. One pile is the expression cards (worth 1 point). The other deck are the duel cards (worth 2 points).
 2. Players take turns choosing a card from the expression deck.
 3. Players solve the expression using an area model.
 4. Players explain how they solved the problem. The player who solved the expression with the least amount of partitioning keeps the expression card.
 5. If both players solved the problem with the same area model, they declare a duel. The players flip over one card from the duel deck. The player who first states an accurate division equation that matches the area model keeps the card.
 6. The player with the most cards/points at the end wins.

In this example, players solve $92 \div 4$. Player 2 would keep the card, since they used two steps to solve the problem.

PLAYER 1			
	10	10	3
4	4×10 40	4×10 40	4×3 12

PLAYER 2		
	20	3
4	4×20 80	4×3 12

Area Model Duel can be modified in a variety of ways. For example, the expression cards can have two-, three-, or four-digit dividends. Create duel cards that have easier expressions if the goal is for students to practice basic facts. Create duel cards that have more complex expressions if the goal is efficiency.

RESOURCE(S) FOR THIS ACTIVITY

8 <table><tr><td>10</td><td>10</td><td>9</td></tr><tr><td>8×10</td><td>8×10</td><td>8×9</td></tr><tr><td>80</td><td>80</td><td>72</td></tr></table>	10	10	9	8×10	8×10	8×9	80	80	72	8 <table><tr><td>20</td><td>9</td></tr><tr><td>8×20</td><td>8×9</td></tr><tr><td>160</td><td>72</td></tr></table>	20	9	8×20	8×9	160	72
10	10	9														
8×10	8×10	8×9														
80	80	72														
20	9															
8×20	8×9															
160	72															
6 <table><tr><td>60</td><td>2</td></tr><tr><td>6×60</td><td>6×2</td></tr><tr><td>360</td><td>12</td></tr></table>	60	2	6×60	6×2	360	12	6 <table><tr><td>30</td><td>30</td><td>2</td></tr><tr><td>6×30</td><td>6×30</td><td>6×2</td></tr><tr><td>180</td><td>180</td><td>12</td></tr></table>	30	30	2	6×30	6×30	6×2	180	180	12
60	2															
6×60	6×2															
360	12															
30	30	2														
6×30	6×30	6×2														
180	180	12														
9 <table><tr><td>70</td><td>6</td></tr><tr><td>9×70</td><td>9×6</td></tr><tr><td>630</td><td>54</td></tr></table>	70	6	9×70	9×6	630	54	9 <table><tr><td>50</td><td>20</td><td>6</td></tr><tr><td>9×50</td><td>9×20</td><td>9×6</td></tr><tr><td>450</td><td>180</td><td>54</td></tr></table>	50	20	6	9×50	9×20	9×6	450	180	54
70	6															
9×70	9×6															
630	54															
50	20	6														
9×50	9×20	9×6														
450	180	54														

$240 \div 3$	$360 \div 6$
$490 \div 7$	$450 \div 5$
$560 \div 8$	$630 \div 9$
$810 \div 9$	$630 \div 7$



Expression cards (on the right) and duel cards (on the left) can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.13

Name: *Make the Quotient*

Type: *Center*

About the Center: This center is a different take on a target-based activity. Here, students are presented with a target quotient and must make a division problem that has a quotient as close to the target as possible. You can make the center more challenging by having students deal themselves six digit cards to select from in order to make the quotient.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), Make the Quotient cards and recording sheet

- Directions:**
1. Students spread out all of the digit cards face up.
 2. Students choose a Make the Quotient card.
 3. Students select from the digit cards to make two problems that satisfy the prompt and write related multiplication problems that they might have used to help them create the problem.
 4. Students record their work and choose a new card.

For example, Marianna pulls “Make the Quotient 12. Use 3 digit cards.” She selects 3, 6, and 3 to make the expression $36 \div 3$, and then she selects 2, 4, and 2 to make the expression $24 \div 2$. She records $36 \div 3 = 12$ and $24 \div 2 = 12$ on her recording page.

RESOURCE(S) FOR THIS ACTIVITY

Make the Quotient				
Directions: Choose a Make the Quotient card. Use digit cards to make two different problems with that quotient. Write a multiplication problem related to the division problem.				
Card	Division Problem 1	Related Multiplication Problem	Division Problem 2	Related Multiplication Problem

A. Make the Quotient 12 Use 3 digit cards __ ÷ __	B. Make the Quotient 15 Use 3 digit cards __ ÷ __
C. Make the Quotient 20 Use 3 digit cards __ ÷ __	D. Make the Quotient 24 Use 3 digit cards __ ÷ __
E. Make the Quotient 16 Use 3 digit cards __ ÷ __	F. Make the Quotient 30 Use 3 digit cards __ ÷ __
G. Make the Quotient 11 Use 3 digit cards __ ÷ __	H. Make the Quotient 13 Use 3 digit cards __ ÷ __



Center cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.14

Name: *Rewrite It and Solve It*

Type: *Center*

About the Center: Rewrite It and Solve It is an opportunity for students to practice rewriting division problems using the inverse.

Materials: Rewrite It and Solve It center cards and recording sheet

Directions: 1. Students select a card at random and record it on their recording sheet.

2. Students rethink the division expression as a missing value multiplication equation and record it in column 2.

3. Students use Think Multiplication to solve and record the quotient in column 3.

Note: This could also be done in a journal rather than on a recording sheet.

TEACHING TAKEAWAY

To save paper, print the recording sheet and put it in a plastic sleeve or laminate for students to write on with dry-erase markers.

RESOURCE(S) FOR THIS ACTIVITY

Rewrite It and Solve It		
Directions: Pull a problem card. Rewrite it as a multiplication problem and find the product. Then, record the division equation.		
Division Problem	Related Multiplication Problem	So...
$120 \div 6 = ?$	$6 \times ? = 120$ $6 \times 20 = 120$	$120 \div 6 = 20$

$240 \div 3$	$360 \div 6$
$490 \div 7$	$450 \div 5$
$560 \div 8$	$630 \div 9$
$810 \div 9$	$630 \div 7$

$360 \div 12$	$480 \div 24$
$450 \div 5$	$600 \div 30$
$640 \div 32$	$320 \div 16$
$750 \div 25$	$396 \div 36$

$176 \div 8$	$372 \div 6$
$497 \div 7$	$455 \div 5$
$576 \div 8$	$648 \div 9$
$837 \div 9$	$644 \div 7$



Center cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.15

Name: *Triangle Cards*

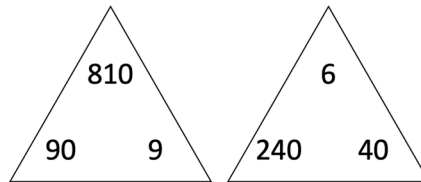
Type: *Center*

About the Center: Triangle fact cards help students develop multiplication and division basic fact recall. These cards focus students on the relationship between the three numbers through either of the operations. This center builds on that by using larger numbers. Any activity that makes use of a basic fact triangle card could be modified for use with these cards.

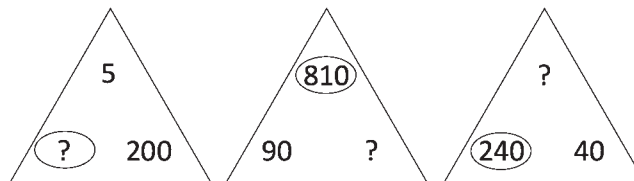
Materials: triangle cards and recording sheet

- Directions:**
1. Students select a triangle card.
 2. The student writes the two multiplication equations the card represents, the two division equations the card represents, and one of the basic facts that all of these connect with.

For example, a student picks the card on the left. They record $9 \times 90 = 810$, $90 \times 9 = 810$, $810 \div 9 = 90$, $810 \div 90 = 9$, and the basic fact $9 \times 9 = 81$.

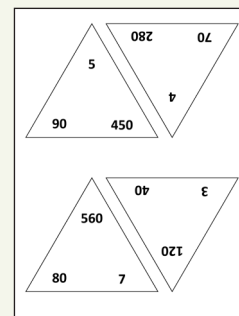


A modified version of this center presents two of the three numbers within the equation. Students must determine what the missing number is and record all of the equations. The product is the circled number, which can also be the missing number.



RESOURCE(S) FOR THIS ACTIVITY

Triangle Cards			
Directions: Choose a triangle card. Write the two multiplication equations and the two division equations that it represents. Then, write a related basic fact.			
Card	Multiplication Equations	Division Equations	Related Basic Fact



Triangle cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 5.16

Name: *Finding the Breaks*

Type: *Center*

About the Center: Breaking apart dividends to Think Multiplication relies on exposure, experience, and practice. Finding the Breaks provides opportunity for each of those. With this center, students choose a division problem and tell how they can decompose the dividend into multiples of the divisor. This center is intended to complement the routine in Activity 5.8 (“Breaking Dividends”). Division problems used with that routine can be applied to this center or you can use the problems from this center for that routine. As with any quality practice opportunity, students should have an opportunity to reflect on the experience and tell how using multiplication can help them solve more complex division problems.

Materials: division expression cards, Finding the Breaks recording sheet

- Directions:**
1. The student pulls a division card and writes it on the recording sheet.
 2. The student identifies how to break apart the dividend into multiples of the divisor so that they can use Think Multiplication to find the quotient.

For example, a student pulls $288 \div 8$. The student breaks 288 into 240 and 48 and uses the inverse to find $240 \div 8$ (30) and $48 \div 8$ (6). She then adds the quotients of 30 and 6 to find the quotient of the original problem of $288 \div 8$.

RESOURCE(S) FOR THIS ACTIVITY

Finding the Breaks			
Directions: Select a division problem. Break apart the dividend into multiples of the divisor. Use multiplication to divide the breaks.			
Problem	Break Apart the Dividend	Divide the Breaks Using Multiplication to Help	So...
$288 \div 8$		$240 \div 8 = 30$ because $8 \times 30 = 240$ $48 \div 8 = 6$ because $8 \times 6 = 48$	$288 \div 8 = 36$ because 5 got 30 and 6 together.



Division expression cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

[illegible]

Partial Quotients Strategy

STRATEGY OVERVIEW: Partial Quotients

What is Partial Quotients? It is a division strategy in which the dividend is broken apart into convenient parts. Those parts are divided separately, then these partial quotients are added together. This strategy is foundational for learning and using the standard algorithm for division. Unlike the standard algorithm, students can find any partial quotient, rather than seek the exact largest partial quotient for a place value.

HOW DOES PARTIAL QUOTIENTS WORK?

Partial Quotients is the distributive property in action. The way the dividend is partitioned is flexible. Let's look at three ways to use Partial Quotients to solve $216 \div 3$.

$ \begin{array}{r} 3 \overline{)216} \\ \underline{-30} \quad 10 \times 3 \\ 186 \\ \underline{-30} \quad 10 \times 3 \\ 156 \\ \underline{-150} \quad 50 \times 3 \\ 6 \\ \underline{-6} \quad 2 \times 3 \\ 0 \\ 72 \times 3 \end{array} $	$ \begin{array}{r} 3 \overline{)216} \\ \underline{-150} \quad 50 \times 3 \\ 66 \\ \underline{-60} \quad 20 \times 3 \\ 6 \\ \underline{-6} \quad 2 \times 3 \\ 0 \\ 72 \times 3 \end{array} $	$ \begin{array}{r} 3 \overline{)216} \\ \underline{-210} \quad 70 \times 3 \\ 6 \\ \underline{-6} \quad 2 \times 3 \\ 0 \\ 72 \times 3 \end{array} $
---	--	---

On the left, Partial Quotients is used by first taking out 10 groups of 3 (30). This is repeated again. Then, the student recognizes they could remove 50 groups of 3 (150), which leaves 6 groups of 3. The partials add to 72. In the middle column, the first group size used is 50. The right column shows the largest partial by place value, 70. This is more efficient, and it is the same as the standard algorithm.

WHEN DO YOU CHOOSE PARTIAL QUOTIENTS?

Partial Quotients always works and is a good choice when Think Multiplication isn't a viable strategy and/or when students are struggling with the standard algorithm. A key to Partial Quotients being a good option is being able to choose partial quotients well, as each of the three earlier examples illustrate. The work shown here, however, is not as efficient. This may be a good beginning, but with experience, the partial quotients need to be grouped more efficiently.

$$\begin{array}{r}
 3 \overline{)216} \\
 \underline{-30} \quad 10 \times 3 \\
 186 \\
 \underline{-30} \quad 10 \times 3 \\
 156 \\
 \underline{-30} \quad 10 \times 3 \\
 126 \\
 \underline{-30} \quad 10 \times 3 \\
 96 \\
 \underline{-30} \quad 10 \times 3 \\
 66 \\
 \underline{-30} \quad 10 \times 3 \\
 36 \\
 \underline{-30} \quad 10 \times 3 \\
 6 \\
 \underline{-6} \quad 2 \times 3 \\
 0
 \end{array}$$

PARTIAL QUOTIENTS: Strategy Briefs for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. These strategy briefs are a tool for doing that. You can include them in parent or school newsletters or share them at parent conferences. They are available for download so that you can adjust them as needed.

Partial Quotients

How It Works: We can remove easy-to-find groups from the dividend until there are no more groups to remove.

- Find a friendly amount to take from the dividend.
- Find a friendly amount to take from what is left.
- Continue removing friendly amounts until nothing is left to remove.
- Find the total number of groups taken away.

The left example shows that 100 is as a friendly amount to remove: 20 groups of 5 are in 100. 100 is removed and 265 is left. Another 100 can be removed, leaving 165. This is continued until nothing is left to remove. Then you add up the number of groups removed: $20 + 20 + 30 + 5 = 73$.

The same process is applied to the right example. However, 350 was a friendly amount removed. Again, there are 73 groups of 5 in 365, so $365 \div 5 = 73$.

When It's Useful: Partial Quotients is almost always useful. It can become difficult to use as the number of digits increases in the divisor or dividend.

Dividend Divisor
 $365 \div 5$

$$\begin{array}{r} 5 \overline{) 365} \\ \underline{-100} \\ 265 \\ \underline{-100} \\ 165 \\ \underline{-150} \\ 15 \\ \underline{-15} \\ 0 \end{array}$$

20×5
 20×5
 30×5
 3×5

73

$365 \div 5 = 73$
Quotient

Dividend Divisor
 $365 \div 5$

$$\begin{array}{r} 5 \overline{) 365} \\ \underline{-350} \\ 15 \\ \underline{-15} \\ 0 \end{array}$$

70×5
 3×5

73

$365 \div 5 = 73$
Quotient

Partial Quotients

How It Works: We can remove easy-to-find groups from the dividend until there are no more groups to remove.

- Find a friendly amount to take from the dividend.
- Find a friendly amount to take from what is left.
- Continue removing friendly amounts until nothing is left to remove.
- Find the total number of groups taken away.

In the example, 20 groups of 52 (1,040) were removed from the dividend (3,684). This is continued until 44 was left to remove. A group of 52 couldn't be removed from 44. So, there are 70 (20 + 20 + 20 + 10) groups of 52 in 3,684 with a remainder of 44.

When It's Useful: Partial Quotients is almost always useful. It can become unwieldy as the number of digits increases in the divisor or dividend.

Dividend Divisor
 $3,684 \div 52$

$$\begin{array}{r} 52 \overline{) 3,684} \\ \underline{-1,040} \\ 2,644 \\ \underline{-1,040} \\ 1,604 \\ \underline{-1,040} \\ 564 \\ \underline{-520} \\ 44 \end{array}$$

20×52
 20×52
 20×52
 10×52

70

$3,684 \div 52 = 70 \text{ R } 44$



These resources can be downloaded at resources.corwin.com/FOF/multiplicativedividewholenumber.

NOTES

MODULE

1

Break Apart to Multiply Strategy

MODULE

2

Halve and Double Strategy

MODULE

3

Compensation Strategy

MODULE

4

Partial Products Strategy

MODULE

5

Think Multiplication Strategy

MODULE

6

Partial Quotients Strategy

MODULE

7

Standard Algorithms for Multiplication and Division

TEACHING ACTIVITIES for Partial Quotients

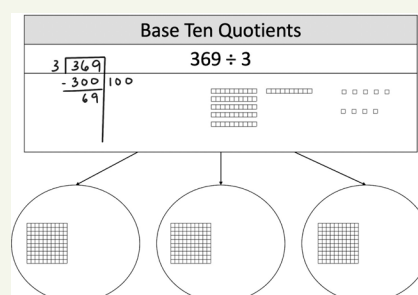
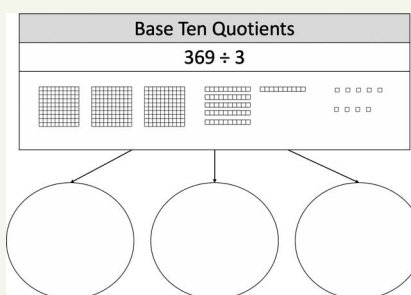
Partial Quotients builds on students' understanding of breaking numbers apart and finding how many groups of a certain size are in a given amount. Partial Quotients lays the foundation for understanding how the standard algorithm works in the future. In this section, we focus on instructional activities for Partial Quotients. The goal is for students to become adept with the strategy and also to consider when the strategy is useful.



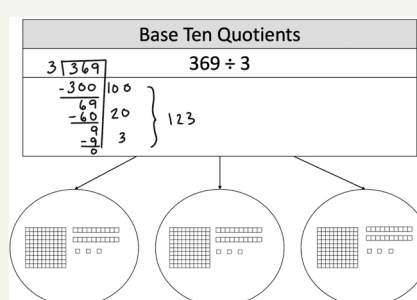
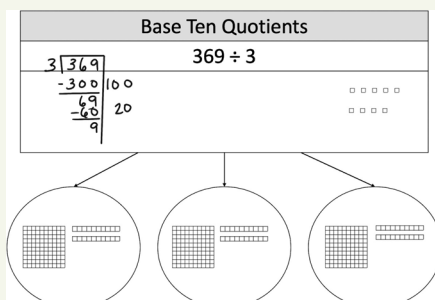
ACTIVITY 6.1 BASE-10 QUOTIENTS

When students first work with division they use manipulatives to find the quotient of basic facts. We want to extend this work as students divide larger numbers. Base-10 blocks are concrete tools to help students understand division. Provide students with expressions, such as $369 \div 3$, and have them represent the dividend using base-10 blocks. Show students how to partition the 369 into 3 equal groups and connect the process with the physical tools to the numerical method.

Start off by partitioning the 3 hundreds into three equal groups. Each group has 1 hundred. When you remove that amount from the dividend, 69 remains.



Then partition the tens into three groups. There are 2 tens in each group; 9 units remain.



Students keep partitioning until nothing remains. By adding the partial quotients together, students then find the final quotient. For this example, there are 123 blocks in each group. It's important to provide students with problems that partition equally but also situations with remainders.



ACTIVITY 6.2 IS IT THE SAME?

Students may tend to overly rely on breaking apart a dividend into tens and ones because those multiplication factors are more comfortable for them. For example, they may solve $66 \div 3$ using $3 \times 10 + 3 \times 10 + 3 \times 2$ to find the quotient rather than consider multiples of 10, such as 3×20 .

$$\begin{array}{r} 3 \overline{)66} \\ -30 \\ \hline 36 \\ -30 \\ \hline 6 \\ -6 \\ \hline 0 \end{array} \quad \begin{array}{l} 10 \times 3 \\ 10 \times 3 \\ 2 \times 3 \end{array}$$

$$\begin{array}{r} 3 \overline{)66} \\ -60 \\ \hline 6 \\ -6 \\ \hline 0 \end{array} \quad \begin{array}{l} 20 \times 3 \\ 2 \times 3 \end{array}$$

This can become quite problematic for problems with larger dividends, such as $549 \div 3$. This teaching activity focuses on establishing that the two approaches yield the same result and that one Partial Quotients strategy may be more efficient than the other. In this activity, you pose a problem such as $66 \div 3$. You discuss with students how it might be broken apart and record their thinking. Then, groups of students work to find the quotient both by using multiples of tens and also with groups of tens (as illustrated above). Explore larger dividends like $549 \div 3$. After students find the quotient, discuss how the options are the same and how they are different. Ask students to determine which approach is more efficient or at least easier to keep track of. Extend the activity by asking students if they think it will always work. Students can create their own problems to find out.



ACTIVITY 6.3 ANOTHER WAY

This activity is an opportunity for students to practice using the area model representation and connecting it to Partial Quotients. For example, a school placed a book order for 989 books. The books were to be evenly distributed to eight teachers. How many books does each teacher receive? Students solve using one strategy. Invite students to review their solutions. They may check their work by comparing answers with a partner who used the same strategy. Students then move on to show the division in another way, now using the alternate strategy. For example, if a student used an area model as their first strategy, then they would solve the problem in another way using the Partial Quotients strategy. At this point, since both strategies have been explored, debrief with the class. Ask students how the area models and partial quotients connected.

Students can discuss how they decided on the parts or factors they would need when solving. As a note, when working with story problems it's important to make sure students are interpreting the remainder correctly. When students are given expressions to solve not in context, check for students' notation of the remainder.

$$\begin{array}{r} 8 \overline{)989} \\ -800 \\ \hline 189 \\ -160 \\ \hline 29 \\ -24 \\ \hline 5 \end{array} \quad \begin{array}{l} 100 \times 8 \\ 20 \times 8 \\ 3 \times 8 \end{array}$$

123 R5

100	20	3	= 123 r5
8 800	20 160	3 24	5



ACTIVITY 6.4 COMPARING PARTIALS

This activity is an opportunity for students to make connections between partial products and partial quotients. Ask the students to fold a paper in half and label the left side “Partial Products” and the right side “Partial Quotients.” Provide students with a multiplication expression and a related division expression. Students then work to solve the problem on the other side of the paper. Then, students come together to share their findings. During the discussion, point out the relationship between the multiplication and division problems.

$\begin{array}{r} 6 \times 58 \\ 58 \\ \times 6 \\ \hline 48 \\ + 300 \\ \hline 348 \end{array}$ (8×6) (50×6)	$\begin{array}{r} 348 \div 6 \\ 6 \overline{)348} \\ - 300 \\ \hline 48 \\ - 48 \\ \hline 0 \end{array}$ 6×50 6×8 (58)
--	---

In this example, students are given the expression 6×58 . Students solve using partial products. Then students move to the other side of the paper and create a division expression $348 \div 6$ based on the equation $6 \times 58 = 348$ on the left side.



ACTIVITY 6.5 PROMPTS FOR TEACHING PARTIAL QUOTIENTS

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking. After students work with the prompt(s), bring the class together to exchange ideas. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once with different numbers. Student work with these prompts could be useful for collecting evidence of understanding.

- How does the basic fact $21 \div 7$ help you solve $2,100 \div 7$?
- How can you use estimation to decide the first partial quotient in $496 \div 8$?
- Samuel says that $3,240 \div 6$ has a partial quotient of 1,600. Natasha disagrees with Samuel. Who is correct? Explain your reasoning.
- Create a story problem for $249 \div 3$.
- Describe an efficient way to solve $1,248 \div 6$ using Partial Quotients.

- Is the quotient of $2,495 \div 5$ over or under 500? Explain your reasoning.
- Create an example of when the Partial Quotients strategy isn't efficient. Justify your thinking.
- Describe to a partner how you use the Partial Quotients strategy. Show an example.
- Chase solved $84 \div 6$ by partitioning the dividend into 60 and 24, then he divided both by 6. Use Partial Quotients to prove if Chase's strategy works in finding a quotient.
- How is using an area model different from the numeric method with the long division symbol (division bracket)? How are they similar?
- Use the digits 1, 3, 5, 7, 9 to create a division problem that divides a four-digit number by a one-digit number with the greatest quotient.
- Use the digits 0, 2, 4, 6, 8, to create a division problem that divides a four-digit number by a one-digit number with the smallest quotient.

NOTES

This image shows a blank sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

PRACTICE ACTIVITIES for Partial Quotients

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the Partial Quotients strategy or are they reverting to previously learned and/or possibly less appropriate strategies such as repeated subtraction? - Are students using the Partial Quotients strategy efficiently? (e.g., Do they work with suitable groups? Or do they take away one group or only 10 groups at a time?) - Do they use the strategy regardless of its appropriateness for the problem at hand? (e.g., $320 \div 8$ is solved best with Think Multiplication.)
Flexibility	- Are students carrying out Partial Quotients in flexible ways? (e.g., Are they able to take away groups of different sizes based on the problem they are working?) - Do they change their approach to or from this strategy as it proves inappropriate or overly complicated for the problem?
Accuracy	- Are students using the Partial Quotients strategy accurately? (e.g., Do they subtract the accurate amount each time?) - Does student recordings of Partial Quotients match what they describe that they are doing? - Are students finding accurate solutions? (e.g., Do they multiply and/or subtract correctly?) - Are they considering the reasonableness of their solutions?* - Are students estimating before finding solutions?*

*This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are problems that have been solved. Correctly worked examples can help students make sense of a strategy and incorrectly worked examples attend to common errors.

Partial Quotients, like Think Multiplication, can lead to the following challenges or errors:

- The student records partial products to start, but the partials go beyond what is needed for the problem at hand.
 - $1,834 \div 22$: notes $22 \times 10 = 220$, $22 \times 100 = 2,200$, and $22 \times 1,000 = 22,000$.
- The student continues to use $\times 10$ facts, without looking for other possible partials.
 - $1,834 \div 22$: 22×10 (220) is subtracted repeatedly, rather than noticing and using 22×20 (440) or 22×30 (660).

3. There is confusion about what “work” is collected to determine the answer.
 - Records a list of partials down the right side of the problem but does not add them together.
4. The student makes subtraction errors, in particular not regrouping when it is necessary.
 - Subtracts 2 – 5 and records 3.

Partial quotients are often obtained by considering multiples of the divisor ($\times 10$, $\times 100$, etc.), but there are other options, which can be highlighted through worked examples. Comparing different worked examples can help students gain insights into efficient use of the Partial Quotients strategy. The prompts from Activity 6.5 can be used for collecting worked examples. Throughout the module are various worked examples that you can use as fictional worked examples. A sampling of additional ideas is provided in the following table.

SAMPLE WORKED EXAMPLES FOR PARTIAL QUOTIENTS		
Correctly Worked Example (make sense of the strategy) What did _____ do? Why does it work? Is this a good method for this problem?	Thomas's work for $1,876 \div 14$: $ \begin{array}{r} 14 \times 10 = 140 \\ 14 \times 100 = 1,400 \\ 14 \overline{) 1,876} \\ \underline{1,400} \quad 14 \times 100 \\ 476 \quad 14 \times 10 \\ \underline{- 140} \quad 14 \times 10 \\ 336 \quad 14 \times 20 \\ \underline{- 280} \quad 14 \times 20 \\ 56 \quad 14 \times 2 \\ \underline{- 28} \quad 14 \times 2 \\ 28 \quad 14 \times 2 \\ \underline{- 28} \quad 14 \times 2 \\ 0 \end{array} $ (134)	Alaina's work for $675 \div 7$: $ \begin{array}{r} 7 \times 10 = 70 \\ 7 \times 50 = 350 \\ 7 \times 90 = 630 \\ 7 \overline{) 675} \\ \underline{- 630} \quad 7 \times 90 \\ 45 \quad 7 \times 6 \\ \underline{- 42} \quad 7 \times 6 \\ 3 \end{array} $ (96 R3)
Partially Worked Example (implement the strategy accurately) Why did _____ start the problem this way? What does _____ need to do to finish the problem?	Autumn's start for $462 \div 6$: $ \begin{array}{l} 462 \div 6 \\ 6 \times 10 = 60 \\ 6 \times 60 = 360 \end{array} $	Mika's start for $625 \div 75$: $ \begin{array}{l} 625 \div 75 \\ 75 \times 2 = 150 \\ 75 \times _ = _ \end{array} $
Incorrectly Worked Example (highlight common errors) What did _____ do? What mistake does _____ make? How can this mistake be fixed?	Matt's work for $7,008 \div 15$: $ \begin{array}{r} 15 \times 10 = 150 \\ 15 \times 100 = 1,500 \\ 15 \overline{) 7,008} \\ \underline{- 1,500} \quad 15 \times 100 \\ 6,508 \quad 15 \times 200 \\ \underline{- 3,000} \quad 15 \times 200 \\ 508 \quad 15 \times 20 \\ \underline{- 300} \quad 15 \times 20 \\ 208 \quad 15 \times 10 \\ \underline{- 150} \quad 15 \times 10 \\ 158 \quad 15 \times 10 \\ \underline{- 150} \quad 15 \times 10 \\ 8 \end{array} $	



ACTIVITY 6.6

Name: *“How Many Are In?”*

Type: *Routine*

About the Routine: When working with multidigit dividends, students are often challenged to determine how many groups of the divisor are in the dividend. Some students will simply start with 10 groups or 100 groups. This approach is acceptable but can create many numbers to keep track of during the process, which increases the likelihood of an error. This routine is a practice opportunity for determining how many groups to take from a dividend. It helps students by providing repetitions and exposes them to others’ thinking.

Materials: Identify different group sizes and dividends as shown.

- Directions:**
1. First reveal the “are in” number, which is 537 in this example.
 2. Give students a moment to think about the multiples of numbers that are close to it.
 3. Then, pose a “how many” number. In this example, students think how many fives are in 537.
 4. Students share their solution and how they arrived at it with a partner and then share out with the group. Some will share there are 100 fives in 537. Some might share 105 fives and some others might find 107 fives. At this time, you don’t need to identify which solution is best, but rather state that each of these is a good idea about how many fives are in 537. Also note that this discussion should not take much longer than a minute or so for the first group. The intention is to get through at least four prompts within the routine. After the first number is discussed, repeat the process with each of the next numbers. If time allows, the routine can be continued with the next set of numbers as shown in the following table.

HOW MANY ____ ARE IN?				
How many	How many	How many	How many	Are in
5s	10s	50s	100s	537

HOW MANY ____ ARE IN?				
How many	How many	How many	How many	Are in
6s	7s	8s	9s	537

Sets 3 and 4 show more advanced approaches to this routine. These would be used with students who are working with multidigit divisors. In these examples, you may choose to provide all four numbers in the set at once, as it may help students see connections and move among them. For example, a student might start with the last prompt (50s) and find there are ten 50s in 537, leading them to know the solution to the first prompt is that there are 50 tens in 537.

HOW MANY ____ ARE IN?				
How many	How many	How many	How many	Are in
10s	20s	25s	50s	537

HOW MANY ____ ARE IN?				
How many	How many	How many	How many	Are in
12s	15s	18s	30s	537



ACTIVITY 6.7

Name: *"Could It Be?"*

Type: *Routine*

About the Routine: Students often complete a computation error and assume that their solution is correct because they have completed the steps. Asking themselves if their solution makes sense or if the solution could be ___ is not always part of their process. Determining if a solution is reasonable and/or accurate plays a significant role in students' fluency. This skill may be most elusive when dividing multidigit numbers. A routine like this provides a practice opportunity and, more importantly, access to others' ideas about how they determine reasonableness and/or accuracy. The unique feature of this routine is that students don't have to find a quotient. Instead, a problem has been completed and students have to determine if the solution is reasonable. This approach simulates what we want students to do as they work through problems.

Materials: Provide two or three completed division problems.

- Directions:**
1. Pose a completed problem.
 2. Have students think about the reasonableness of the solution included with the problem without solving it.
 3. Ask students to share their thinking with a partner.
 4. After partners talk, the whole group comes together to share their ideas about the reasonableness of the solution.
 5. Pose a second problem and repeat the process. As time allows, present a third (or fourth) problem.

COULD IT BE?		
$676 \div 6 = 121$	$785 \div 9 = 105$	$348 \div 6 = 58$

In this example, students first talk about $676 \div 6 = 121$. The quotient is not reasonable. Students might say so because $100 \times 6 = 600$ and $20 \times 6 = 120$, so 120 would be at least 720 and the dividend is 676. Another student might connect $66 \div 6$ to $660 \div 6$ because both are friendly and the latter, $660 \div 6$, is close to $676 \div 6$. The second prompt, $785 \div 9 = 105$, cannot be reasonable because 9×100 is 900 and 785 is much less than 900. The last prompt is correct but students don't have to find the exact solution to prove that it is reasonable. Instead, they might suggest that $6 \times 50 = 300$, that 6×58 will be some more than 300, and that 348 is some more than 300.

Include incorrect answers that are still reasonable estimates (e.g., $512 \div 12 = 42$). Students can follow up the discussion with finding an exact answer. This makes the point that looking for reasonableness does catch many mistakes, but not always.



ACTIVITY 6.8

Name: *Quotient Hunt*

Type: *Game*

About the Game: This is a game with division problems that can be solved with the Partial Quotients strategy. Players choose which numbers to work with, divide, and then “hunt” for a description that matches their quotient.

Materials: *Quotient Hunt* board, one for each group

- Directions:**
1. Each player chooses a divisor and a dividend. Players can only use the numbers once.
 2. The player uses Partial Quotients to find the quotient.
 3. Once the player has the quotient, they hunt on the game board to find the descriptions that match their quotient. The player writes the quotient in that box. If more than one box is a true statement for their quotient, they can pick which box to use (but can only take one box on a turn).
 4. Player 2 repeats the same steps.
 5. The player who gets four spaces arranged in a square wins the game.

For example, player 1 chooses the largest number from the dividend row and the smallest number from the divisor row. Player 1 created the largest quotient. Player 1 then places their quotient (1,800) in the space on the board that is labeled “Largest Quotient.” However, the player could have placed their quotient in the space that is labeled “4-Digit Quotient With No Remainder.”

Quotient Hunt

Directions: Choose a dividend and a divisor. Solve using Partial Quotients. Write your quotient in the box it describes. If your quotient describes more than one box you can choose a box but you can only select one box per turn. You can write a quotient in more than one box. Be the first player to make a square with four spaces.

Dividend							Divisor							
60	200	240	300	324	450	900	3,600	2	3	4	5	6	7	8

Odd Quotient	Quotient With No Remainder	Even Quotient	Quotient That Is a Multiple of 10
Smallest Quotient	2-Digit Quotient 54	3-Digit Quotient With No Remainder	Largest Quotient 1800
4-Digit Quotient With No Remainder	Quotient With an Even Remainder	3-Digit Quotient	Quotient With an Odd Remainder
2-Digit Quotient With No Remainder	Quotient That Is a Multiple of 3	Quotient That Is a Multiple of 2	4-Digit Quotient

Player 1
 $3600 \div 2$

$$\begin{array}{r} 2 \overline{) 3600} \\ \underline{-2000} \\ 1600 \\ \underline{-1600} \\ 0 \end{array} \quad \begin{array}{r} 1000 \\ 800 \\ \hline 1800 \end{array}$$

Player 2
 $324 \div 6$

$$\begin{array}{r} 6 \overline{) 324} \\ \underline{-300} \\ 24 \\ \underline{-24} \\ 0 \end{array} \quad \begin{array}{r} 50 \\ 4 \\ \hline 54 \end{array}$$

RESOURCE(S) FOR THIS ACTIVITY

Quotient Hunt

Directions: Choose a dividend and a divisor. Solve using Partial Quotients. Write your quotient in the box it describes. If your quotient describes more than one box you can choose a box, but you can only select one box per turn. You can write a quotient in more than one box. Be the first player to make a square with four spaces.

Dividend								Divisor							
60	200	240	300	324	450	900	3,600	2	3	4	5	6	7	8	

Odd Quotient	Quotient With No Remainder	Even Quotient	Quotient That Is a Multiple of 10
Smallest Quotient	2-Digit Quotient	3-Digit Quotient With No Remainder	Largest Quotient
4-Digit Quotient With No Remainder	Quotient With an Even Remainder	3-Digit Quotient	Quotient With an Odd Remainder
2-Digit Quotient With No Remainder	Quotient That Is a Multiple of 3	Quotient That Is a Multiple of 2	4-Digit Quotient



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.9

Name: *End of the Lane*

Type: *Game*

About the Game: *End of the Lane* is an opportunity for students to practice the Partial Quotients strategy. Students make division problems and move forward or backward depending on the accuracy of their solution and if it has a remainder.

Materials: digit cards (0–10) or a 10-sided die, *End of the Lane* game board, calculator, scratch paper

- Directions:**
1. Both players begin at the starting line of their own lane.
 2. Both players generate four digits to make a division problem.
 3. Each player solves their problem using Partial Quotients.
 4. If the quotient has a remainder, the player moves two spaces. If the problem has no remainder, the player moves forward one space.
 5. Both players check their opponent's work with a calculator. If the calculator proves the player's quotient is incorrect, the player moves back one space.
 6. Players make a new problem and continue.
 7. The first player to reach the end of their lane wins.

RESOURCE(S) FOR THIS ACTIVITY

End of the Lane

Directions: Place your chip on Start. Create a division expression. Write the expression on the game board. Solve using Partial Products. If the product has a remainder, move forward two spaces. If the product does not have a remainder, move forward one space. Your opponent will check your work with a calculator. If you are incorrect, you move back one space. Be the first to reach the Finish.

Player 1

Start					
					Go Back 1 Space
Finish					

Player 2

Start					
					Go Back 1 Space
Finish					

1	2	3	4
5	6	7	8
9	1	2	3
4	5	6	7
8	9	0	0



Digit cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.10

Name: *Quotient Connect*

Type: *Game*

About the Game: This game is an engaging way to practice division using partial quotients. It is played similarly to the classic board game, *Boggle*, as students find quotients through connected digits.

Materials: *Quotient Connect* game board; digit cards (0–9), playing cards (queens = 0, aces = 1, kings and jacks removed), or division cards; paper for recording work

- Directions:**
1. Players make a game board by randomly placing digits in each grid as shown in the example.
 2. Players use digit cards to make a division problem. As an alternative, you can provide cards with division problems on them.
 3. Both players solve the division problem.
 4. Players look for the quotient on the game board they made by finding adjacent digits (similar to the classic game *Boggle*).
 5. Players get 1 point for each example they find. The first player to reach a set goal (e.g., 10 points) or the most points after five problems wins.

For example, two players made the problem $189 \div 9$. They both found the quotient to be 21. Player 1 found 21 two times, earning 2 points as shown. Player 2 found 21 three times, earning 3 points.

Player 1

Quotient Connect

7	5	4	9	8
4	2	3	8	3
1	2	9	6	4
5	5	9	3	2
6	1	1	3	7

Player 2

Quotient Connect

1	3	6	4	8
9	1	2	3	8
5	6	4	4	1
3	8	7	2	1
3	7	9	4	8



Digit cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.11

Name: : *Remainder Tally*

Type: *Game*

About the Game: *Remainder Tally* is a quick-and-easy game for practicing Partial Quotients division. Players earn points based on the remainders in the quotients they find. You can change the dividend for endless game possibilities. Note that you can also have students use two digit cards to create two-digit divisors (as appropriate).

Materials: two decks of digit cards (2–9), one *Remainder Tally* game board per player

Directions: The game begins with setting a dividend. In the example, the dividend is 500.

1. Each round, both players choose a digit card from the deck.
2. Players use their digit cards as divisors of the set dividend.
3. Each player finds the quotient of the dividend and their divisor using Partial Quotients.
4. Players earn points equivalent to the remainder of their quotient.
5. Players shuffle digit cards and repeat the actions for round 2.
6. After five rounds, each player adds their points.
7. The player with the most points wins.

For example, Jaeona draws a 7 from the deck and uses partials to determine the answer to $500 \div 7$:

$$7 \times 70 = 490$$

$$7 \times 1 = 7$$

$$500 \div 7 = 71, r 3$$

Her remainder is 3. She records 3 as her score.

RESOURCE(S) FOR THIS ACTIVITY

Remainder Tally			
Directions: Pull a number. Divide the dividend by the number you roll. Solve the problem with Partial Quotients and show your thinking. Record your remainder as a score for the round. Add your score for five rounds. The player with the lowest score wins.			
Dividend for This Game:			
	My Number (Divisor)	Solved With Partial Quotients	My Score (Remainder)
1			



Digit cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.12

Name: *Division Bump*

Type: *Game*

About the Game: *Division Bump* reinforces the skill of dividing by 10, 100, and 1,000. The goal of the game is to cover the most spaces on the game board.

Materials: *Division Bump* cards and game board, about 30 counters each (in two different colors)

- Directions:**
1. Each player has a set of color counters.
 2. Player 1 picks a card from the pile and divides the expression.
 3. Player 1 finds the quotient on the *Division Bump* game board and places their counter on that space.
 4. Player 2 repeats the steps above.
 5. If the opponent already has a counter on the space the player may bump the counter.
 6. If a player already has their own counter on the space, the player may stack their counter on top. This space is now locked and can no longer be bumped.
 7. The game ends when all the numbers are covered. The winner is the player with the most counters on the game board.

For example, player 1 picks up the card $90 \div 10$. The player places their counter on the 9. Player 2 picks up the card $600 \div 60$ and places their counter on the 10. Player 1 gets $8,000 \div 1,000$ and places their counter on the 8. Next, player 2 picks up $7,000 \div 700$. Player 2 now has the opportunity to lock their space by stacking their counter on their previous counter.

1	10	1000	100	9	0
1000	2	10	8	1	10
10	1000	3	100	1000	100
100	7	10	4	1	10
6	100	1000	0	5	100

RESOURCE(S) FOR THIS ACTIVITY

$1,000 \div 1$	$2,000 \div 2$
$100 \div 1$	$200 \div 2$
$10 \div 1$	$20 \div 2$
$100 \div 100$	$200 \div 200$
$100 \div 0$	$200 \div 0$

Division Bump					
Remember that players have different color counters. When a number is covered on the game board, the player who placed the counter on that space cannot place another counter on that space. If a player already has their counter on the space, they may stack their counter on top of the other. This space is now locked and cannot be bumped. The player who has the most counters on the game board wins.					
1	10	1000	100	9	0
1000	2	10	8	1	10
10	1000	3	100	1000	100
100	7	10	4	1	10
6	100	1000	0	5	100



Division cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.13

Name: : *Tic-Tac-Take*

Type: *Game*

About the Game: This game is a take on the classic *Tic-Tac-Toe* in which players try to get three markings in a row. The twist in this version is that players can take another player's spot. The game is an opportunity to practice dividing using partial quotients.

Materials: *Tic-Tac-Take* game board, calculator

- Directions:**
1. Player 1 (X) chooses an expression from the game board.
 2. Both players work to solve that expression using partial quotients.
 3. If both players have the same answer, player 1 marks (X) that spot.
 4. If players disagree about the quotient, the division is checked with a calculator.
 5. The player with the correct answer takes the spot. If neither player is correct, the space is crossed off and cannot be played in the game.
 6. Player 2 (O) repeats the steps above.
 7. The winner is the first player with three in a row.

TEACHING TIP

Like most activities, you can swap out the division expressions on the downloadable, offering a different practice experience each time.

RESOURCE(S) FOR THIS ACTIVITY

Tic-Tac-Take Directions: Players take turns selecting an expression on the game board. Both players solve the problem using Partial Quotients. If both players' answers are the same, the player who selected the expression places their mark on that spot. If players disagree, they check the problem with a calculator. The player who is correct takes the spot. The first player to get three in a row wins.		
$728 \div 8$	$592 \div 6$	$684 \div 6$
$373 \div 3$	$490 \div 5$	$963 \div 7$
$129 \div 4$	$288 \div 9$	$824 \div 2$



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.14

Name: *How Many Are In?*

Type: *Center*

About the Center: This center is an independent practice opportunity that reinforces Activity 6.6 ("How Many Are In?"). This center affords students practice dividing with different friendly divisors. It is used here to give you an example of how any routine might be used as a center.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), How Many Are In? recording sheet

- Directions:**
1. Students use digit cards to make a dividend. The number of digit cards used can be adjusted so that students work with appropriate division problems.
 2. Students determine how many of a given group size are in that dividend. You can direct students to use the same "how many" number for each new dividend they create (left example).
 3. Students explain how they know.

For example, a student creates 394 with digit cards. The student determines that there are seven groups of 50 in 394 and explains that 7×50 is 350. This justification makes sense but he could have also said that seven groups of 5 tens is 350 or 7×5 is 35 so 7×50 is 350. You can ask students to find the same "how many" number for each new dividend they create as shown in the left example. The middle example shows how you can ask students to find how many of different amounts are in the same divisor. The right example shows different divisors and four-digit dividends.

Divisor	Dividend	How Many
50s	394	7 because $7 \times 50 = 350$
50s	816	
50s	213	
50s	990	
50s	504	

Divisor	Dividend	How Many
25s	378	
10s	378	
100s	378	
25s	378	
10s	378	

Divisor	Dividend	How Many
12s	2,405	
15s	2,816	
30s	6,590	
60s	3,437	
80s	7,598	

RESOURCE(S) FOR THIS ACTIVITY

How Many Are In?

Directions: Use digit cards to make a divisor and record it in the "are in" column. Determine how many of a given amount are in the divisor and tell how you know.

How Many	Are In	How Do You Know?
50s	362	7 because $7 \times 50 = 350$



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.15

Name: *Estimate and Divide*

Type: *Center*

About the Center: Estimating a solution before finding it is a strategy for determining reasonableness. When used with division, it helps students determine if the partial quotients they are finding make sense and if they are “on the right track.” In this center, students create a division problem with digit cards, estimate the quotient, find the quotient, and determine if they made a good estimate.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), Estimate and Divide recording sheet, calculator (optional)

Directions: Each student:

1. makes a division problem with digit cards.
2. estimates the quotient.
3. finds the quotient using Partial Quotients.
4. confirms that their quotient is correct with a calculator (optional).
5. determines if they made a good estimate.

TEACHING TAKEAWAY

Be sure that students understand there is no one correct estimate or one correct way to estimate.

RESOURCE(S) FOR THIS ACTIVITY

Estimate and Divide			
Directions: Make a division problem. Estimate the quotient. Find the quotient. Tell if you made a good estimate.			
Division Problem	Estimate	Actual Quotient	Was It a Good Estimate?



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.16

Name: *The Largest Quotient,
The Smallest Quotient*

Type: *Center*

About the Center: This center reinforces partial quotients while also focusing on number sense and reasoning. It can be used as a game where students earn points for finding the largest or smallest quotient within a group. You can adjust this center by having students use the same digit cards to create four different problems, two with the largest quotient and two with the smallest quotient. Doing so gives students better insight into how quotients change as problems change. It also helps them in case they don't successfully make the largest or smallest quotient on their first try.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed); Largest Quotient, Smallest Quotient recording sheet

- Directions:**
1. Students pull four digit cards to make a three-digit divisor and a one-digit dividend. Note that this can be adjusted to make division problems aligned to what students are working with.
 2. Students arrange the digit cards to make the largest quotient and find the quotient using partial quotients.
 3. Students then rearrange the same cards to make a division problem with the smallest quotient, again finding the actual quotient with the Partial Quotients strategy.

RESOURCE(S) FOR THIS ACTIVITY

The Largest Quotient, The Smallest Quotient	
Largest Quotient	Smallest Quotient
Division Problem:	Division Problem:
Show how you divided:	Show how you divided:
Quotient:	Quotient:

1	2	3	4
5	6	7	8
9	1	2	3
4	5	6	7
8	9	0	0



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 6.17

Name: **Changing Dividends Make for Changing Quotients**

Type: **Center**

About the Center: Quotients change when the dividend changes and the divisor remains the same. With whole numbers, the quotient increases as the dividend increases. This activity is an opportunity for students to experience and observe how quotients change. When discussed, it can help students reinforce this idea and advance both number sense and sense of reasonableness.

Materials: digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), Changing Dividends Make for Changing Quotients recording sheet

- Directions:**
1. Students select digit cards to make a division problem. For example, students working with three-digit dividends divided by one-digit divisors would select four cards.
 2. Students arrange the cards to make a division problem and find the quotient.
 3. Students rearrange the digits in the dividend while the divisor remains unchanged. The student then predicts if the new quotient will be greater or less than the first quotient.
 4. Students find the quotient of the new problem.
 5. This can be repeated for a third quotient or a new problem can be created and the steps can be repeated.

In this example (recording sheet not used), the student creates $214 \div 8$ and then makes a new problem $142 \div 8$, predicting that the quotient will be less.

$$\begin{array}{r} 214 \div 8 \\ 8 \overline{)214} \\ \underline{-160} \quad (8 \times 20) \\ 54 \\ \underline{-48} \quad (8 \times 6) \\ 6 \end{array}$$
$$\begin{array}{r} 142 \div 8 \\ \text{The quotient} \\ \text{will be less} \\ 8 \overline{)142} \\ \underline{-80} \quad (8 \times 10) \\ 62 \\ \underline{-56} \quad (8 \times 7) \\ 6 \end{array}$$
$$214 \div 8 = 26 \text{ R}6 \quad 142 \div 8 = 17 \text{ R}6$$

RESOURCE(S) FOR THIS ACTIVITY

Changing Dividends Make for Changing Quotients	
Directions: Use digit cards to make a division problem. Find the quotient. Rearrange the digits in the dividend to make a new problem, keeping the divisor the same. Predict if the new quotient will be greater or less than the first. Show how you divided in each problem.	
First Division Problem:	Second Division Problem:
Show how you divided:	Predict if the new quotient will be greater or less than the quotient from the first problem.
	Show how you divided:
Quotient:	Quotient:

1	2	3	4
5	6	7	8
9	1	2	3
4	5	6	7
8	9	0	0



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

[illegible]

Standard Algorithms for Multiplication and Division

STRATEGY OVERVIEW:

Standard Algorithms

What is a Standard Algorithm? Algorithms vary from country to country, and even region to region. In the United States, the standard algorithm for whole number multiplication involves using Partial Products, beginning with the ones place and working to the left (larger) place values. Notations do not define the algorithm (Fuson & Beckmann, 2012–2013). So, both of the following examples illustrate the standard algorithm, one showing each partial product and the other using shorthand notations.

RECORDING EACH PARTIAL PRODUCT IN ITS OWN ROW	USING SHORTHAND NOTATION
$ \begin{array}{r} 347 \\ \times \quad 7 \\ \hline 49 \\ 280 \\ 2,100 \\ \hline 2,429 \end{array} $	$ \begin{array}{r} 347 \\ \times \quad 7 \\ \hline 2,429 \end{array} $

The U.S. standard algorithm for division is a Partial Quotients approach but rather than use any partial quotient, the largest possible partial is expected. It is an iterative process of seeking the largest possible partial quotient for each place value, subtracting, and repeating for the next smaller place value.

HOW DO U.S. STANDARD ALGORITHMS WORK?

Standard algorithms are based on place value and the properties. This fact is more visible in the longhand version rather than the shorthand, which is why Partial Products and Partial Quotients are important processes to connect to the standard algorithms, as illustrated in the 347×7 example. Division is perhaps the hardest algorithm to understand, possibly because you must consider more than one place value as the dividend relates to the divisor. In the following example of $7,420 \div 35$, you are looking not just at the thousands place but also at the hundreds place. The language associated with the algorithm is digit based (the italicized words are typically not stated). Using the italicized words has the benefits of connecting and strengthening number sense (e.g., Ask, “How many 35s are in 7,400?” rather than “How many 35s are in 74?”). Because many students have a fragile understanding of multiples, it is easier for them to answer the digit-based version.

EXAMPLE	WHAT IS SAID
$ \begin{array}{r} 212 \\ 35 \overline{) 7,420} \\ \underline{- 7,000} \\ 420 \\ \underline{- 350} \\ 70 \\ \underline{- 70} \\ 0 \end{array} $	How many 35s in 7 <i>thousand</i>? (0) How many 35s in 74 <i>hundreds</i>? (2) Record 2 hundreds. Multiply 35×200. (7,000) Subtract that quantity from the total. <i>Repeat:</i> How many 35s in 42 <i>tens</i>? (1) Record 1 ten. Multiply 35×10. (350) Subtract that from the remaining total. (70) <i>Repeat:</i> How many 35s in 70? (2) Record 2. Multiply 35×2. (70) Subtract that from the remaining total. (0)

WHEN DO YOU CHOOSE STANDARD ALGORITHMS?

While standard algorithms always work, they are not always the best choice. A standard algorithm is needed in the following situations:

- Numbers in the problem do not lend themselves to a mental method.
- Numbers in the problem do not lend themselves to a convenient written option (e.g., compensation).
- You don't know an alternate method.
- You want to check an answer having used a different method.

Perhaps because standard algorithms are in state standards, a major part of our own learning, and applicable to all problems, there can be a hurry to get to them and significantly more time devoted to learning them than their reasoning alternatives. This is a mistake. A rush to the standard algorithm and memorizing procedures undermines students' confidence and may cause math anxiety, negatively impacting achievement (Boaler, 2015a, b; Jameson, 2013; Ramirez, Shaw, & Maloney, 2018). Note that "fluently multiply" does not mean "use the standard algorithm adeptly." It means you use the standard algorithm adeptly *when it is the most efficient option*. Such flexibility is at the heart of ensuring students develop fluency.

MODULE

1

Break Apart to Multiply Strategy

MODULE

2

Halve and Double Strategy

MODULE

3

Compensation Strategy

MODULE

4

Partial Products Strategy

MODULE

5

Think Multiplication Strategy

MODULE

6

Partial Quotients Strategy

MODULE

7

Standard Algorithms for Multiplication and Division

STANDARD ALGORITHMS: Strategy Briefs for Families

It is important that families understand the strategies and know how they work so that they can be partners in the pursuit of fluency. These strategy briefs are a tool for doing that. You can include them in parent or school newsletters or share them at parent conferences. They are available for download so that you can adjust them as needed.

Standard Algorithm for Multiplication

How It Works: The algorithm is a digit-based procedure for multiplying multidigit factors.

1. Multiply the digit in the ones place of the bottom factor by each digit in the top factor.
2. Continue to multiply digits by place value from right to left. Record products of each new place value below the products of the previous place value.
3. Add the products of each place value.

The left example shows a problem with no regrouping. The products of 1 and each digit in the top factor are recorded. The products of the second digit, which represents tens, are recorded on the second line. The two products are added for a sum of 9,982. So, $322 \times 31 = 9,982$.

The right example shows a problem with regrouping: $2 \times 6 = 12$. The ones are recorded and the 10 is grouped to be added to the product of the tens place. 2×4 tens is 8 tens and 1 more ten is 9 tens recorded as shown. Finding the products of 7 tens requires additional regrouping. The two products are added for a sum of 24,912. So, $346 \times 72 = 24,912$.

When It's Useful: The algorithm becomes more useful as the number of digits increases in each factor.

$$\begin{array}{r} 322 \\ \times 31 \\ \hline 322 \\ + 9,660 \\ \hline 9,982 \end{array}$$

$$\begin{array}{r} 346 \\ \times 72 \\ \hline 692 \\ + 24,220 \\ \hline 24,912 \end{array}$$

Standard Algorithm for Division

How It Works: The algorithm is a digit-based procedure for dividing.

1. Determine how many groups of the divisor are in an amount of the dividend. In the following example, there are no (whole number) groups of 35 in 3. So, it is expanded to 36. There is 1 group of 35 in 36.
2. 1×35 is 35. Note that the 1 is in the hundreds place so it is 1 hundred times 35, creating 3,500.
3. 3,500 is subtracted from 3,687 with a difference of 187.
4. There are no (whole number) groups of 18, so a 0 is recorded and 187 is considered.
5. There are 5 groups of 35 in 187. $5 \times 35 = 175$.
6. In this example, 3,687 divided by 35 is 105 with a remainder of 5.

When It's Useful: The algorithm becomes more useful as the number of digits increases in the dividend or divisor.

$$\begin{array}{r} \text{Dividend} \quad \text{Divisor} \\ 3,687 \div 35 \\ \hline 105 \text{ R } 12 \\ 35 \overline{) 3,687} \\ \underline{- 3,500} \\ 187 \\ \underline{- 175} \\ 12 \end{array}$$



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

NOTES

TEACHING ACTIVITIES for Standard Algorithms

Explicit strategy instruction for standard algorithms has been a pillar of elementary school. The use of base-10 blocks and other tools has helped students to see how each functions conceptually. You likely have many resources for teaching the steps of each algorithm. The teaching activities offered here do not focus on that task but do not suggest that it isn't important. Instead, these tasks focus on making connections between strategies and the algorithm, uncovering errors within an algorithm, and exposing students to thinking about when an algorithm is or isn't useful. Keep in mind that determining reasonableness plays a role in fluency and it is critically important as students carry out algorithms because procedural and computational errors are quite possible.



ACTIVITY 7.1 CONNECTING PARTIAL PRODUCTS AND THE ALGORITHM

This instructional activity is an opportunity for observing the relationship between Partial Products and the standard algorithm for multiplication. It is somewhat unique in that it does not require students to find a product or to complete the steps for finding a product. Instead, students complete a problem using both approaches and are told that both are completed correctly. Then, students are charged with observing the two approaches and finding the many ways that they are the same and different.

$\begin{array}{r} 45 \\ \times 27 \\ \hline 800 \\ 280 \\ 100 \\ \hline 35 \\ \hline 1,215 \end{array}$	$\begin{array}{r} 45 \\ \times 27 \\ \hline 35 \\ 280 \\ 100 \\ \hline 800 \\ \hline 1,215 \end{array}$	$\begin{array}{r} 45 \\ \times 27 \\ \hline 315 \\ 900 \\ \hline 1,215 \end{array}$
---	---	---

This example shows how the activity can be used with 2 two-digit factors. The example on the left shows partial products carried out beginning with the tens place of the factors and the middle example shows partials beginning with the ones place of the factors. The right example shows the standard algorithm. Students are likely to notice the relationship between the two Partial Product approaches because the products are in reverse order. The relationship between the partials and the standard algorithm may be less obvious. Students will eventually see that 315 is the sum of 35 and 280 and that 900 is the sum of 800 and 100. This can buoy a discussion as to why this may be (because the algorithm combines steps or partials) and if it will always happen (yes!).

To continue the activity, you can pose additional completed problems for students to observe and comment on. Then, you can extend the activity by charging students to solve a problem on their own with both processes.



ACTIVITY 7.2 CONNECTING PARTIAL QUOTIENTS AND THE ALGORITHM

Activity 7.2 is directly related to the previous activity. Here, students are presented with a division problem correctly completed with Partial Quotients and the standard algorithm. Students are prompted to identify how the work is similar and different. It too should be complemented with multiple examples to reinforce the relationship. This instructional activity may be even more powerful with division because it shows the utility and friendliness of partial quotients and the streamlined work of the standard algorithm. In the example, the standard algorithm is not accompanied by all of the “scratch work” to determine products, differences, and other computations. This is the case because the divisor is somewhat friendly. That would not be the case if the divisor was something like 29 or 37.

$$\begin{array}{r}
 28 \overline{) 687} \text{ R}15 \\
 \underline{-240} \quad 24 \times 10 \\
 447 \\
 \underline{-240} \quad 24 \times 10 \\
 207 \\
 \underline{-120} \quad 24 \times 5 \\
 87 \\
 \underline{-48} \quad 24 \times 2 \\
 39 \\
 \underline{-24} \quad 24 \times 1 \\
 15
 \end{array}$$

This example shows a three-digit dividend divided by a two-digit divisor. Naturally, students would start with more simplistic problems for working with division but the activity would be carried out the same way. This activity and Activity 7.1 are effective ways to make sense of the shorthand used in the standard algorithms for multiplication and division, as compared to solely connecting standard algorithms to base-10 blocks or other visuals.

Also note that with this activity, you can pose one problem at a time for students to observe and discuss as described in Activity 7.1. Yet another approach would be to provide different completed problems as different “stations” around the room and have partners move from station to station observing the problems and recording their thinking to be discussed during a whole group conversation after the rotations are completed.



ACTIVITY 7.3 FIND THE ERROR

Error analysis is when students analyze an example of completed work, looking specifically for the error made. Error analysis is a higher-level thinking skill because it requires a deeper understanding of the skill or concept to be able to identify (and correct) errors. Errors could include computation miscalculation or missing step(s) misconceptions.

This activity requires students to analyze work examples, find errors, and explain how to rectify the mistakes. “Find the Error” can be done as a whole class discussion. This allows students to collaborate with one another and listen to different explanations from their peers. Display one problem at a time for students to find errors and explain the mistake. Provide students a mix of multiplication or division problems. The following examples show different types of errors that students would work through.

COMPUTATION ERROR

$$\begin{array}{r}
 67 \\
 \boxed{6} \overline{) \boxed{3} \boxed{9} \boxed{4}} \\
 \underline{- \boxed{3} \boxed{6} \boxed{0}} \\
 \boxed{4} \boxed{4} \\
 \underline{- \boxed{4} \boxed{2}} \\
 \boxed{2}
 \end{array}$$

GREATEST FACTOR ERROR

$$\begin{array}{r}
 31 \\
 \boxed{4} \overline{) \boxed{1} \boxed{2} \boxed{9}} \\
 \underline{- \boxed{1} \boxed{2} \boxed{0}} \\
 \boxed{0} \boxed{9} \\
 \underline{- \boxed{0} \boxed{4}} \\
 \boxed{5}
 \end{array}$$

An alternative to this activity is to provide students with problems that contain errors and some that don't, so students work to distinguish between the two. "Find the Error" can transition from a whole class activity to independent practice or centers. See also the section on Worked Examples, where a series of examples are provided.



ACTIVITY 7.4 TO ALGORITHM OR NOT TO ALGORITHM

Fluency with standard algorithms includes not only how to use an algorithm but *when* to use an algorithm. It is critical that we empower students to think critically about when they need this tool. This activity aims to do just that. Pose a collection of multiplication or division problems to small groups of students. Have the students think about which problems call for an algorithm and which don't. Students must show their strategies for problems that don't call for an algorithm. Conversely, students must tell why a certain problem is best suited for the algorithm.

use algorithm	Don't use algorithm
$340 \div 39$	$486 \div 12$
$782 \div 16$	$800 \div 40$
$516 \div 55$	$650 \div 25$

In this example, the teacher posed a list of six different problems. Students worked with partners to sort and record those problems. They determined that the three problems on the right didn't require an algorithm. For $486 \div 12$, they recognized $48 \div 12$ and were able to break 486 into $480 + 6$. They explained (*not shown*) that to solve $650 \div 25$, you could break apart 650 into $100 + 100 + 100 + 100 + 100 + 100 + 50$ and figure out how many 25s were in each. For $800 \div 40$, a student shared that you could use Think Multiplication using the basic fact $4 \times ? = 8$.



ACTIVITY 7.5 PROMPTS FOR TEACHING STANDARD ALGORITHMS

Use the following prompts as opportunities to develop understanding of and reasoning with the strategy. Have students use representations and tools to justify their thinking, including base-10 models, number lines, arrays, and so on. After students work with the prompt(s), bring the class together to exchange ideas. These could be useful for collecting evidence of student understanding. Any prompt can be easily modified to feature different numbers (e.g., three-digit or four-digit numbers) and any prompt can be offered more than once if modified.

- Find the product of $8 \times 2,021$ with a strategy and the algorithm. Tell how the two are similar and different.
- Create a problem that would be best solved with the standard algorithm. Tell why this problem is a good choice for solving with the standard algorithm.
- Tasha and Michael solved 24×47 but came up with different products. Who is correct? Explain how you know.

Tasha	Michael
$\begin{array}{r} 24 \\ \times 47 \\ \hline 168 \\ 960 \\ \hline 1,128 \end{array}$	$\begin{array}{r} 24 \\ \times 47 \\ \hline 168 \\ 96 \\ \hline 264 \end{array}$

- Devon said when you multiply a two-digit number by a one-digit number, you get a two-digit product. Natasha disagreed with Devon and said you get a three-digit product. Who is correct? How do you know?
- Create a story problem for $249 \div 3$. Solve using the standard algorithm, explaining how the steps connect to the story.
- Cierra says the product of 296×126 will be less than 35,000. Without solving the problem, explain how you know if Cierra is correct or not.
- Dylan divides two numbers and the quotient is 150. What could be the two numbers Dylan used?
- How is the Partial Products strategy for multiplication similar to and different from the standard algorithm for multiplication? Use a problem to show your thinking.
- Create and solve a division problem with the standard algorithm. Describe what is happening in each part of the problem.
- Create a division problem that would be best solved with the standard algorithm and a division problem that doesn't need the standard algorithm for solving. Tell how you know when you will not use the standard algorithm.
- Think about the standard algorithm for multiplication (or division). What do you think is a common error that people make when using it?

PRACTICE ACTIVITIES for Standard Algorithms

Fluency is realized through quality practice that is focused, varied, processed, and connected. The activities in this section focus students' attention on how this strategy works and when to use it. The activities are a collection of varied engagements. The discussion you facilitate after an activity or the reflection prompts you attach to it should help students think about what they did mathematically, how they reasoned about the activity, and when the math they did (namely the strategy) might be useful. Debriefing should also help students see how the practice activity connects to recent instruction or how the strategy connects to other strategies they know. Game boards, recording sheets, digit cards, and other required materials are available as online resources for you to download, possibly modify, and use. As students work with activities, look for how well they are acquiring the strategy and assimilating it into their collection of strategies.

FLUENCY COMPONENT	WHAT TO LOOK FOR AS STUDENTS PRACTICE THIS STRATEGY
Efficiency	- Are students using the algorithm when another strategy is clearly more efficient? (e.g., 45×12 can be solved with the algorithm but also aligns with Halve and Double as well as Partial Products.) - Are students using the algorithm efficiently? (i.e., Are they choosing problems that are best for solving with the algorithm?) - Do they use the algorithm regardless of its appropriateness for the problem at hand? (e.g., Do they use it for 400×6 or $250 \div 50$?)
Flexibility	- Are students using the algorithm when it is unnecessary? (e.g., $900 \div 300$) - Do students change their approach to or from the algorithm as it proves inappropriate or overly complicated for the problem? (e.g., A student abandons $455 \div 5$ after finding 9 tens, recognizing that the dividend is $450 + 5$ and that they can use Think Multiplication more efficiently.)
Accuracy	- Are students using the algorithm accurately? (e.g., Are they accurately recording or keeping track of place values?) - Are students finding accurate solutions when using the algorithm? - Are they considering the reasonableness of their solutions?* - Are students estimating before they use the algorithm?*

* This consideration is not unique to this strategy and should be practiced throughout the pursuit of fluency with whole numbers.

WORKED EXAMPLES

Worked examples are opportunities for students to analyze the thinking of another student. Worked examples include correct and incorrect examples. While algorithms always work, the enactment of established steps and notations can lead to mistakes along the way. Worked examples focused on common errors can help students develop an understanding of why that step does not work.

The prompts from Activity 7.5 can be used for collecting examples. Throughout the module are various examples that you can use as worked examples. The following table shows common errors and a sample worked example with that error. Because standard algorithms are error prone, estimating and checking for reasonableness are very important. When you analyze worked examples, you can begin by asking, "Is this answer reasonable?" If the answer is "no," then ask, "What is a reasonable answer?" Of course, there are wrong answers that turn out to be reasonable, which is why it is also good to check for accuracy, either by rechecking the steps or using the inverse operation with the answer.

COMMON ERROR	WORKED EXAMPLE
Ignores internal zeros (multiplication)	$\begin{array}{r} 907 \\ \times 8 \\ \hline 776 \end{array}$ $\begin{array}{r} 6,003 \\ \times 8 \\ \hline 4,824 \end{array}$
Has difficulty with zeros (division)	$\begin{array}{r} 35 \\ 3 \overline{)915} \\ -900 \\ \hline 15 \end{array}$ $\begin{array}{r} 20R5 \\ 12 \overline{)2,405} \\ -24 \\ \hline 005 \end{array}$
Adds regrouped number before multiplying a place value (multiplication)	$\begin{array}{r} 64 \\ 257 \\ \times 7 \\ \hline 5,639 \end{array}$
Loses track of place value (multiplication)	$\begin{array}{r} 316 \\ \times 23 \\ \hline 948 \\ + 632 \\ \hline 1,580 \end{array}$
Loses track of place value, especially when zeros are involved (division)	$\begin{array}{r} 15R16 \\ 28 \overline{)2,956} \\ -2,800 \\ \hline 156 \\ -140 \\ \hline 16 \end{array}$



ACTIVITY 7.6

Name: *“Over/Under”*

Type: *Routine*

About the Routine: Determining reasonableness when using standard algorithms is critically important. In this routine, students determine if a product or quotient will be over or under (more or less) than a given number.

Materials: Identify a target number for comparison and prepare a few multiplication or division expressions that would likely be solved with the standard algorithm.

- Directions:**
1. Share the over/under number.
 2. Pose the first expression.
 3. Have students estimate the product or quotient of each problem.
 4. Students share their ideas with partners.
 5. The class then discusses their approaches and reasoning. After ideas are shared, the exact solution can be found and compared to the over/under number.
 6. Continue the routine with the next expression.

Over/Under 50		
$355 \div 7$	$248 \div 33$	$2445 \div 59$

In this example, students might argue that $7 \times 50 = 350$, so the first quotient must be more than 50. The middle expression might be reasoned that 30×8 is 240 so the quotient is about 8, which is much less than 50. The last example shows how the routine might evolve to less friendly divisors. Here, students might suggest that $2400 \div 60$ is about 40 and because they increased the divisor, the answer may be even smaller than 40, but certainly under 50.

Over/Under 25		
$287 \div 7$	$306 \div 3$	$84 \div 2$

The first two dividends in this example are less obviously connected to basic facts. However, students can still use multiplication to reason about quotients. In the first expression, one might think that $40 \times 7 = 280$, so then the quotient is clearly more than 25. Similarly, in the middle expression, $3 \times 100 = 300$ so that quotient must be more than 25 as well. The third may be thought of differently as students might contend that half of 84 is 40 rather than thinking about multiplication. It is a good reminder that although Think Multiplication is the focus of the routine in these examples, students can (and will) offer other, viable thoughts and reasons.



ACTIVITY 7.7

Name: *“That one”*

Type: *Routine*

TEACHING TAKEAWAY

With some guidance, students can help create problems for games and routines by writing them on sticky notes or index cards.

About the Routine: This routine helps students consider when to use an algorithm. It is intended as a practice opportunity after instruction of the concept (see Activity 7.4 “To Algorithm or Not to Algorithm”). Problems from the other routines as well as those throughout this book work great for this routine.

Materials: Prepare three or four expressions for students to consider.

- Directions:**
1. Pose a few problems to students.
 2. Have students discuss with a partner which problems are good candidates for solving with an algorithm and which are not.
 3. Have partners explain their decisions in a class discussion. Be sure to avoid implying that any problem should or must be solved with an algorithm.

That One		
600×3	599×6	789×4

In this example, students might identify 789×4 as best solved with an algorithm, as the first could be solved by thinking about the basic fact 6×3 and the second could be solved with compensation by thinking 600×6 and then taking a group of 6 away. Students could also argue that 789×4 might be solved with a compensation strategy ($800 \times 4 - 44$), in which case they would respond that “none” of these are best solved with an algorithm.



ACTIVITY 7.8

Name: *Call Your Shot*

Type: *Game*

About the Game: *Call Your Shot* offers an opportunity to practice with multiplication or division in an engaging way. In this game, players create situations and then try to make a problem that matches the condition. Players earn a point if they are unable to do so, with an object of the game being not to earn points.

Materials: two sets of digit cards (0–9) or playing cards (queens = 0, aces = 1, kings and jacks removed), list of conditions, math journal (optional)

- Directions:**
1. Player 1 identifies a condition (calls their shot).
 2. Both players pull three or more digit cards to make a problem.
 3. A player gets a point if they can't make a problem to fit the condition.
 4. Player 2 then identifies a condition and both players attempt to make a problem to meet the condition.
 5. The first player to get 5 points loses.

For example, player 1 says make a quotient between 0 and 50. Both players pull digit cards. Player 1 makes $391 \div 7$ and player 2 makes $332 \div 8$. Player 1 would get a point because their quotient is 55 r 6. Player 2 would not get a point because they met the condition.

Here are some examples for three-digit divided by one-digit problems. Players can also create their own conditions.

EVEN QUOTIENT	ODD QUOTIENT	QUOTIENT BETWEEN 0 AND 50	QUOTIENT BETWEEN 60 AND 90
Quotient with a 0 in it	Quotient with a 5 in it	Quotient greater than 70	Quotient less than 25
Quotient with a remainder	Quotient without a remainder	Two-digit quotient	Three-digit quotient
Quotient where each digit is an even number	Quotient where each digit is an odd number	Quotient with an odd remainder	Quotient with an even remainder



ACTIVITY 7.9

Name: *A List of Ten*

Type: *Game*

About the Game: In terms of procedural fluency, knowing when to use an algorithm is just as important as knowing how to use an algorithm. Yet students don't have frequent opportunities to consider when to use an algorithm. *A List of Ten* is a game for considering when an algorithm is useful and when it isn't. It is the perfect complement to instructional Activity 7.4 ("To Algorithm or Not to Algorithm").

Materials: *A List of Ten* game cards and game board

- Directions:**
1. Expression cards are shuffled and placed face down.
 2. Players take turns selecting a card and determining if the expression is best solved with or without the algorithm. Note that if a player can tell their opponent why an expression is better solved without an algorithm, they can steal the card from their opponent.
 3. The player records the equation in the appropriate column.
 4. The first player to get a list of 10 wins.
 5. After the game, both players solve each problem on their game board.

The following example shows the first three problems that player 1 flipped over. Player 2 says that $132 \div 6$ could be thought of as $120 \div 6 + 12 \div 2$ and steals the problem. Player 1 will mark it off of their page and player 2 will add it to the left column on their recording sheet.

Problems solved WITHOUT the STANDARD ALGORITHM	Problems solved WITH the STANDARD ALGORITHM
1. $640 \div 80 =$	1. $388 \div 18 =$
2.	2. $132 \div 6 =$

RESOURCE(S) FOR THIS ACTIVITY

A List of Ten

Directions: Players take turns flipping over cards and deciding if the problem is best solved with or without the standard algorithm. Players can steal problem cards put in the algorithm list if they can explain how a different strategy could be used to solve the problem.

Problems to solve WITHOUT the STANDARD ALGORITHM	Problems to solve WITH the STANDARD ALGORITHM
1.	1.
2.	2.

$132 \div 2$	$258 \div 3$
$640 \div 80$	$78 \div 3$
$160 \div 4$	$344 \div 4$
$118 \div 2$	$388 \div 18$
$171 \div xxx$	$132 \div 6$



Game cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.10

Name: *For Keeps*

Type: *Game*

About the Game: *For Keeps* is an opportunity to practice multiplication in an engaging way that also helps students develop reasoning about numbers. Players find products and determine if they want to keep the product to count toward their final score. The game can be adjusted easily so that students practice division by choosing to keep or discount quotients.

Materials: *For Keeps* game board; 10-sided dice or playing cards (queens = 0, aces = 1, kings and jacks removed), digit cards (0–9)

- Directions:**
1. Each player generates their own four numbers to make 2 two-digit factors.
 2. Players find the product of the two factors and decide if they want to keep the product as the score for the round or if they don't want to keep it for a score. Once they decide (before the next problem), the decision is final and the product can't be moved later.
 3. Players play a total of four rounds yet can only keep two of those rounds. At the end of the fourth round, each player adds the two scores that they kept.
 4. The player with the higher score wins.

In the example, the player first made 22×31 for a product of 682 and didn't keep it. The second product of 1,156 was kept. The player thinks they can do better than their third-round score of 700 in the fourth round. No matter what product is made in the fourth round, the player will have to keep it. The sum of the two kept scores (1,156 and the fourth round) will be the player's score.

Round	Numbers Created		Product For Keeps	Product NOT Kept
1	22	31		682
2	68	17	1,156	
3	14	50		700
4				
	Sum of Keeps			

Show your thinking

RESOURCE(S) FOR THIS ACTIVITY

For Keeps

Directions: Make 2 two-digit numbers. Find the product and decide if you want to keep the product as a score or not keep it. You can only keep two of the four rounds. After the fourth round, add the two scores you kept. The player with the higher score wins.

Round	Numbers Created	Products for Keeps	Products NOT Kept
1			
2			
3			
4			
	Sum of Keeps		



Digit cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.11

Name: *The V Cover Up*

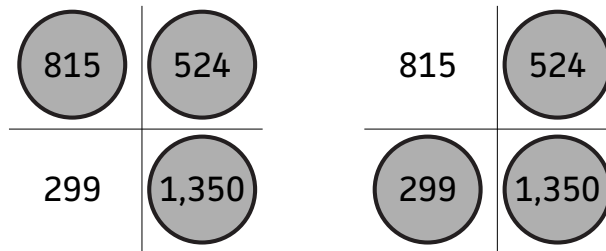
Type: *Game*

About the Game: *The V Cover Up* is an opportunity to practice multiplication and division in an engaging way. Players solve problems in order to cover spaces on their game board so they make a three-space V. You can have students practice both operations, as the spaces represent products and quotients of the cards, or you can remove one operation for isolated play.

Materials: *The V Cover Up* game board, counters, expression cards, calculators

- Directions:**
1. Players take turns pulling multiplication and division cards and finding the product or quotient.
 2. The player covers the product or quotient they find on the game board. Their opponent checks their accuracy with a calculator.
 3. If the product or quotient is already taken or the player computes incorrectly, the player loses their turn.
 4. The first player to make a three-space V wins. (Or keep playing, and whoever gets the most Vs when the board is filled or time is up is the winner.)

The following image shows how to make a three-space V. The V can be oriented in any way.



RESOURCE(S) FOR THIS ACTIVITY

9×64	19×60
6×72	4×232
54×25	3×682
18×25	2×512
42×24	5×376

The V Cover Up				
Directions: Take turns choosing a card. Find the product or the quotient and cover the space on the board. Your partner will check your work with a calculator. Be the first to make a V.				
576	784	1,008	928	1,024
207	1,360	307	1,082	82
432	486	1,140	2,046	1,880
49	1,072	815	524	967
516	807	299	1,350	176
1,199	119	814	754	306



Expression cards and this game board can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.12

Name: *Missing Numbers With Multiplication Algorithms*

Type: *Center*

About the Center: Missing Numbers With Multiplication Algorithms helps students practice finding and correcting multiplication errors. This center can be complemented with a writing component. For example, after students complete a Missing Numbers card, they can be expected to write about how they found the missing digit or to justify how they know their solution is correct.

Materials: prepared Missing Number problem cards (directions follow)

- Directions:**
1. Students select a Missing Number problem card.
 2. Students work to find the Missing Number(s) in the algorithm.

To make Missing Numbers problem cards, create a problem and then simply change out one or more digits with a question mark. The template provided is for two-digit by one-digit multiplication. Additional templates featuring two-, three-, and four-digit factors are available for download.

$$\begin{array}{r} \boxed{?} \boxed{?} \\ \times \quad \boxed{?} \\ \hline \boxed{4} \boxed{8} \end{array}$$

$$\begin{array}{r} \boxed{?} \boxed{?} \\ \times \quad \boxed{5} \boxed{8} \\ \hline \boxed{1} \boxed{2} \boxed{7} \boxed{6} \end{array}$$

$$\begin{array}{r} \boxed{?} \boxed{?} \boxed{?} \boxed{?} \\ \times \quad \boxed{?} \\ \hline \boxed{1} \boxed{7} \boxed{0} \boxed{3} \boxed{2} \end{array}$$

RESOURCE(S) FOR THIS ACTIVITY

$\begin{array}{r} \square \square \\ \times \quad \square \\ \hline \square \square \end{array}$	$\begin{array}{r} \square \square \\ \times \quad \square \\ \hline \square \square \end{array}$
$\begin{array}{r} \square \square \\ \times \quad \square \\ \hline \square \square \end{array}$	$\begin{array}{r} \square \square \\ \times \quad \square \\ \hline \square \square \end{array}$



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.13

Name: *Missing Digits With Division Algorithms* **Type:** *Center*

About the Center: This is a problem-based activity for practicing the standard algorithm for division. Students work to find the missing components (digits) in problems that have already been solved.

Materials: Prepare Missing Digit problem cards.

Directions:

1. Students select a Missing Digit problem card.
2. Students work to find the Missing Digit(s) in the algorithm.

To make Missing Digit problem cards, create a problem and then simply change out one or more digits with a question mark. A template is provided for three-digit by one-digit division. In this example, the teacher made the problem $142 \div 2 = 71$ and removed a few of the digits. The student then worked to find the missing digits to be 4, 0, 2, then 0 (from top to bottom).

$$\begin{array}{r} 71 \\ 2 \overline{) 142} \\ \underline{- 140} \\ 02 \\ \underline{- 02} \\ 0 \end{array}$$

RESOURCE(S) FOR THIS ACTIVITY

$\square \overline{) \square \square \square}$ $\underline{\square \square \square}$ $\square \square$ $\underline{\square \square}$ $\square \square$	$\square \overline{) \square \square \square}$ $\underline{\square \square \square}$ $\square \square$ $\underline{\square \square}$ $\square \square$
$\square \overline{) \square \square \square}$ $\underline{\square \square \square}$ $\square \square$ $\underline{\square \square}$ $\square \square$	$\square \overline{) \square \square \square}$ $\underline{\square \square \square}$ $\square \square$ $\underline{\square \square}$ $\square \square$



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.14

Name: *Partials to Algorithms*

Type: *Center*

About the Center: Connecting algorithms to other strategies, especially Partial Products and Partial Quotients, helps students make sense of the procedure. In this center, students are given a problem completed with Partial Products. Students then have to rewrite the problem using the standard algorithm.

Materials: Partials to Algorithms cards and recording sheet

- Directions:**
1. Students select a card that shows a problem completed with partial products.
 2. Then, students solve the problem again with the standard algorithm.
 3. Students then tell which approach they thought was the better choice for the problem (and why).

RESOURCE(S) FOR THIS ACTIVITY

$$46 \times 27$$

$$\begin{array}{r} 40 \times 20 = 800 \\ 40 \times 7 = 280 \\ 6 \times 20 = 120 \\ 6 \times 7 = 42 \\ \hline 1,242 \end{array}$$

$$382 \times 9$$

$$\begin{array}{r} 300 \times 9 = 2,700 \\ 80 \times 9 = 720 \\ 2 \times 9 = 18 \\ \hline 3,438 \end{array}$$

$$91 \times 19$$

$$\begin{array}{r} 90 \times 10 = 900 \\ 90 \times 9 = 810 \\ 1 \times 10 = 10 \\ 1 \times 9 = 9 \\ \hline 1,729 \end{array}$$

$$16 \times 13$$

$$\begin{array}{r} 10 \times 10 = 100 \\ 10 \times 3 = 30 \\ 6 \times 10 = 60 \\ 6 \times 3 = 18 \\ \hline 208 \end{array}$$

Partials to Algorithms

Directions: Select a card that shows a problem solved with partials. Solve the problem with the standard algorithm. Tell which approach was the better choice for the problem.

Problem	Solved With the Standard Algorithm	Which Was the Better Choice? Why?



Cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.15

Name: **Making Connections**

Type: **Center**

About the Center: A Frayer model is a graphic organizer that helps students make connections between concepts and ideas. This center uses an adaptation of the classic Frayer model by focusing on strategies and algorithms. In this center, students are given a problem. They solve the problem with different strategies or approaches and then comment on which was most useful for the problem. You can use this center with multiplication or change out the prompts asking students to represent the problem with a diagram or write a word problem that goes with the expression in the center.

Materials: Multiple Methods graphic organizer

- Directions:**
1. Students are given or select a division problem (recorded in the center of the organizer).
 2. Students apply three different strategies to the problem and then explain which method they thought was the best fit for that problem.

The example shows a student who completed $154 \div 7$.

Solve it with Think Multiplication $\begin{array}{l} 7 \times ? = 154 \\ 7 \times 20 = 140 \\ 7 \times 2 = 14 \\ 20 + 2 = 22 \end{array}$	Solve with Partial Quotients $\begin{array}{r l} 7 & 154 \\ -140 & 20 \\ \hline & 14 \\ -14 & 2 \\ \hline & 0 \\ & 22 \end{array}$
Solve with the standard algorithm $\begin{array}{r} 22 \\ 7 \overline{)154} \\ -14 \downarrow \\ \hline 14 \\ -14 \\ \hline 0 \end{array}$	Tell which approach was best for this problem Think multiplication was best for this problem. I knew that $7 \times 20 = 140$ so my product had to be close to 20. Then I just had to find the other missing factor.

$154 \div 7$



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY 7.16

Name: *Correct the Error*

Type: *Center*

About the Center: As students complete the steps in an algorithm, they make errors. Often students continue on without noticing. In other instances, they don't look back to see where an error occurred if their answer doesn't align with what they anticipated. This center is an opportunity to practice looking for errors. It can help students develop a habit of looking over their work when completing procedures.

Materials: Correct the Error recording sheet

- Directions:**
1. Students review the problem in the first column of the recording sheet.
 2. Students find the error and tell what it was in the second column of the recording sheet.
 3. Students work to solve the problem correctly on the last column.

The problem	What is the error?	Solve
$\begin{array}{r} 31\overline{)136} \text{ R2} \\ 4\overline{)136} \\ -120 \\ \hline 6 \\ -4 \\ \hline 2 \end{array}$	The error was $136 - 120$ does not equal 6. $136 - 120 = 16$.	$\begin{array}{r} 34 \\ 4\overline{)136} \\ -120 \\ \hline 16 \\ -16 \\ \hline 0 \end{array}$

RESOURCE(S) FOR THIS ACTIVITY

Correct the Error		
Directions: Review the problem in the first column. Find the error. Write down what the error is in the second column. In the third column, solve the problem correctly.		
The Problem	What is the Error?	Solve
$\begin{array}{r} 17 \\ 3\overline{)528} \\ -300 \\ \hline 228 \\ -210 \\ \hline 18 \end{array}$		
$\begin{array}{r} 52 \\ 5\overline{)245} \\ -255 \\ \hline 10 \\ -10 \\ \hline 0 \end{array}$		
$\begin{array}{r} 802 \\ 10\overline{)8020} \\ -800 \\ \hline 20 \\ -20 \\ \hline 0 \end{array}$		



This resource can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

[illegible]

PART 3

PUTTING IT ALL TOGETHER

Developing Fluency

FLUENCY IS . . .

How might you finish a sentence that begins “Fluency is . . .”? One way is “using procedures efficiently, flexibly, and accurately.” Another option is “important.” Or maybe fluency is “an equity issue.” All of these are true statements. This section is the capstone of this book on fluency with multiplication and division of whole numbers. If students learn a strategy in isolation and never get to practice choosing when to use it, they will not become truly fluent. Part 3 focuses on learning to *choose* strategies. The following lists reflect subsets of the Seven Significant Strategies.

MULTIPLICATION “MUST KNOW” STRATEGIES	DIVISION “MUST KNOW” STRATEGIES
• Break Apart to Multiply • Halve and Double • Compensation • Partial Products	• Think Multiplication • Partial Quotients

Once students know more than one strategy, they need opportunities to practice choosing strategies. In other words, you do not want to wait until they have learned all strategies and the standard algorithms before they begin choosing from among the strategies they know. For example, if you are teaching Compensation, you may have likely already taught Break Apart to Multiply. So, it is time to use a Part 3 activity to help students decide *when* they will use each strategy. Then, add a new strategy to students’ repertoire and return again to activities that focus on choosing a strategy. This iterative process continues through the teaching of standard algorithms, as students continue to accumulate methods for multiplying and dividing whole numbers. Along the way, it is important that students also learn that sometimes there is more than one good way to solve a problem, and other times, one way really stands out from the others.

CHOICE

Choosing strategies is at the heart of fluency. After a strategy is learned, it should always be an option for consideration. Too often students feel like when they have moved on to a new strategy, they are supposed to use only the new one, as though it is more sophisticated or preferred by their teacher. But, the strategies are additive—they form a collection from which students can select in order to solve the problem at hand.

Once students know how and why a strategy works, they need to figure out when it makes sense to use the strategy. That is when you need questions such as these:

- When do you/might you use a Break Apart strategy for multiplication?
- When do you/might you use a Compensation strategy for multiplication?
- When do you/might you use Partial Products? Partial Quotients?
- When do you/might you use Think Multiplication?
- When do you/might you use a standard algorithm?

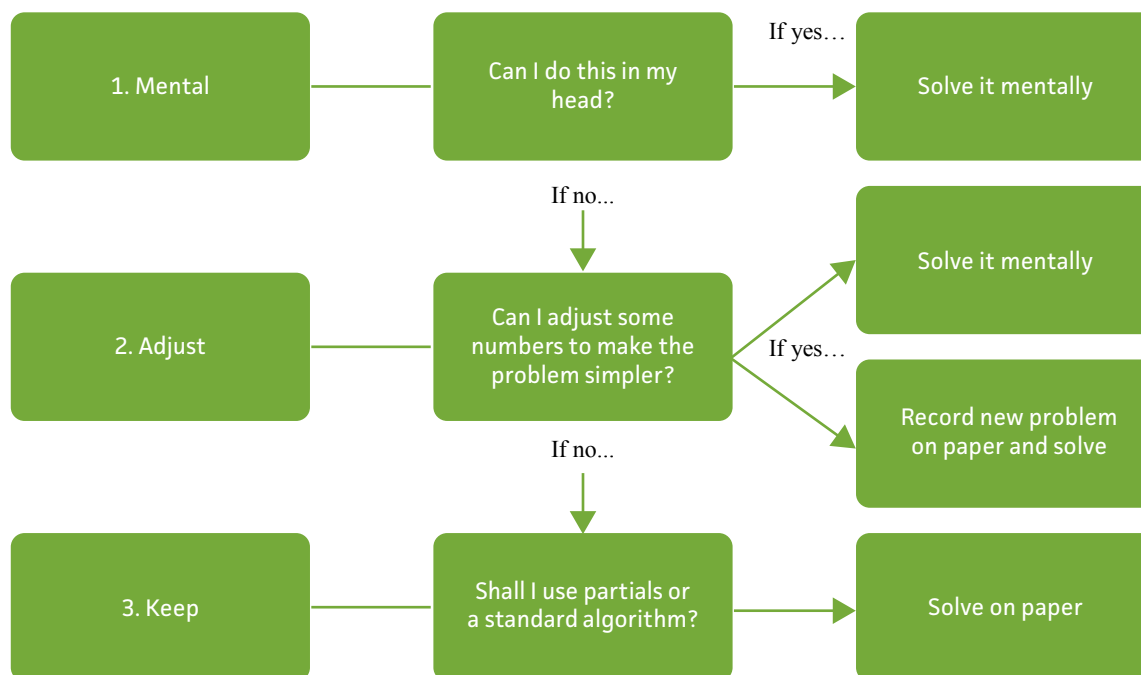
- When do you/might you not use _____ strategy/algorithm?
- Which strategies do you think of first when you see a problem?
- Which strategies do you choose only after other options don't work?
- Which strategies do you avoid?

The first six prompts can be mapped to the Part 2 modules (and asked during instruction on those strategies, as well as during mixed practice). As students are exploring the Compensation strategy, for example, include regular questioning about when it is a good fit and when it is not. The full set of questions is essential to the development of fluency. Activities are woven throughout the modules to help identify when a strategy works best and when it doesn't. However, those activities don't necessarily bring other strategies into consideration. Part 3 focuses on this notion of putting it all together, which includes attending to metacognition.

METACOGNITION

As students become more proficient with the strategies they have learned and when to use them, they make decisions as to what they will do for a given problem. Taking time up front to make a good choice can save time in the enactment of a not-efficient strategy. We can help students with this reasoning by sharing a metacognitive process. This could be a bulletin board or a card taped to students' desks. We can remind students that as we work, we make good choices about the methods we use. This is *flexibility* in action.

METACOGNITIVE PROCESS FOR SELECTING A STRATEGY



ASSESSING FLUENCY

Traditionally, fluency assessment has focused on speed and accuracy. While accuracy does matter, it is only one-third of what it means to be fluent. As you assess fluency, you want to intentionally look for each of the three fluency components and the six Fluency Actions. Flexibility and efficiency can be observed as students engage in meaningful activities. There is no need to stop and give a test.

OBSERVATION TOOLS

Observation tools can help us focus on the “neglected” components of fluency and serve as a way to communicate with students and their parents about what real fluency looks like. Three examples are shared here.

See Chapter 7 (pp. 154–175) of *Figuring Out Fluency* for more information about assessing fluency.

Student: _____ Date: _____

Problem(s): _____

Fluency Actions Checklist

Procedural Fluency Actions	Evident?			Comments
1. Selects appropriate strategy	Yes	No	Not Observed	
2. Solves in a reasonable amount of time	Yes	No	Not Observed	
3. Trades out or adapts a strategy	Yes	No	Not Observed	
4. Applies strategy to a new problem type	Yes	No	Not Observed	
5. Completes steps correctly	Yes	No	Not Observed	
6. Gets correct answer	Yes	No	Not Observed	

Comments: _____

Multiplication Strategy Observation Tool

Student Name	Multiplication Strategies				Algorithm
	Break Apart to Multiply	Halve and Double	Compensation	Partial Products	

Division Strategy Observation Tool

Student Name	Division Strategies		Algorithm
	Think Multiplication	Partial Quotients	



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

GRADING ASSIGNMENTS AND ASSESSMENTS

All of your efforts to teach strategies and to teach students to choose efficient strategies must be continued through assessments. Too often tests are graded for accuracy only. Instead, provide a fluency score using a rubric, such as the one that follows.

FOUR-POINT FLUENCY RUBRIC

BEGINNING 1	DEVELOPING 2	EMERGING 3	ACCOMPLISHED 4
Knows one algorithm or strategy but continues to get stuck or make errors.	Demonstrates efficiency and accuracy with at least one strategy/algorithm, but does not stop to think if there is a more efficient possibility.	Demonstrates efficiency and accuracy with several strategies/algorithms, and sometimes selects an efficient strategy, although still figuring out when to use and not use a strategy.	Demonstrates efficiency and accuracy with several strategies/algorithms and is adept at matching problems with efficient strategies (knowing when to use each and when not to).

With an eye on fluency and how to assess it, select fluency activities from Part 3 based on your students' needs and what type of activity you are seeking.

FLUENCY ACTIVITIES

The nine activities in this section focus on all components of fluency, providing students with the opportunity to practice choosing an appropriate strategy, enact that strategy, and reflect on the efficiency of the strategies selected.



ACTIVITY F.1

Name: "Strategize First Steps"

Type: Routine

About the Routine: This routine involves sharing how to *start* a problem. It is at this first step that students are selecting a strategy. In its quickest version, that is the only step that is ever done. Simply ask, "Which step first and why?" A second option begins the same way but after students have shared ideas for first steps, they get to choose which first step they like and finish the problem. Then, answers can be shared and students can reflect on which first step turned out to be a good one (and why).

Materials: list of three or four problems on the same topic, but that lend to different reasoning strategies

Directions: This routine involves showing a series of problems, one at a time, like a Number Talk. But it differs in the fact that the tasks you selected *do not* lend to the same strategy. Here is the routine process:

1. Ask students to mentally determine their first step (only) and signal when they're ready (since it is only the first step, they only need a few seconds).
2. Record first-step ideas by creating a list on the board.

(Continued)

(Continued)

3. Discuss which first steps seem reasonable (or not).
4. Repeat with two to four more problems, referring to the list created from the first problem and adding to the list if/when new strategies are shared.
5. Conclude the series with a discussion: When will you use ____ strategy?

Possible problem sets for this routine include the following:

TWO-DIGIT BY ONE-DIGIT MULTIPLICATION	TWO-DIGIT BY ONE-DIGIT DIVISION
$59 \times 8 =$	$55 \div 5 =$
$35 \times 6 =$	$65 \div 3 =$
$52 \times 4 =$	$59 \div 7 =$
$27 \times 9 =$	$99 \div 4 =$
$67 \times 3 =$	$72 \div 6 =$
THREE-DIGIT BY ONE-DIGIT MULTIPLICATION	THREE-DIGIT BY ONE-DIGIT DIVISION
$185 \times 4 =$	$185 \div 6 =$
$490 \times 5 =$	$490 \div 2 =$
$258 \times 7 =$	$258 \div 4 =$
$315 \times 6 =$	$315 \div 7 =$
$207 \times 8 =$	$207 \div 5 =$
TWO-DIGIT BY TWO-DIGIT MULTIPLICATION	FOUR-DIGIT BY TWO-DIGIT DIVISION
$28 \times 13 =$	$6,420 \div 12 =$
$68 \times 59 =$	$8,650 \div 25 =$
$71 \times 35 =$	$1,360 \div 40 =$
$86 \times 57 =$	$9,810 \div 15 =$

Let's say you were using the list for two-digit multiplication. Here are some examples of the first steps you might hear for the first problem, 28×13 :

"Change 28 to 30"

"Break apart both factors"

"Think 28×10 is 280"

You can layer in strategy names at this point. For example, label the first idea as Compensation, the second as Break Apart (or Partial Products), and so on. Then, by consistent use of the language, students should begin to say, "I used Compensation and . . ." For each problem, some of the same strategies will be named again, and new ideas might be added. If an idea doesn't come up, you can ask about it directly or just move to the next problem. Be mindful not to imply that it is the preferred approach. As noted earlier, if students are going to solve the problem, allow them to choose from the ideas that were first stated. They can later share if they stayed with their original first step or switched out to one of the ideas from their classmates.



ACTIVITY F.2

Name: *“M-A-K-E a Decision”*

Type: *Routine*

About the Routine: Mental-Addjust-Keep-Expressions is a routine that, like other routines, can function for assessment purposes. Project for students an illustration of the metacognitive process, like the one discussed earlier in the Part 3 overview. If a problem can be solved mentally, there is no need to use a written method such as a standard algorithm. Students tend to dive in without stopping to think if they can solve a problem mentally. There are two ways to use the tasks in this routine. One is to show one problem at a time, each time asking, “Which way will you solve it?” Or you can show the full set and ask, “Which ones might you solve mentally? Adjust and solve? Keep as-is and solve?” The former is used below.

Materials: Prepare a set of three to six expressions.

- Directions:**
1. Explain that you are going to display a problem. Students are not to solve it but simply “M-A-K-E a Decision: Mental, Adjust and solve, or Keep and solve on paper?”
 2. Display the first problem and give students about 10 seconds to decide.
 3. Use a cue to have students share their choice (one finger = mental, two fingers = adjust, three fingers = keep and solve on paper).
 4. Ask a student who picked each decision to share why (and how).
 5. Repeat step 3, giving students a chance to change their minds.
 6. Repeat for the next problem.
 7. At the end of the set, ask, “What do you ‘see’ in a problem that leads to you doing it mentally? Adjusting? Keep and use paper?”

Possible problem sets to use with this routine include the following:

TWO-DIGIT BY ONE-DIGIT MULTIPLICATION		TWO-DIGIT BY ONE-DIGIT DIVISION	
$31 \times 8 =$	$17 \times 9 =$	$59 \div 5 =$	$65 \div 4 =$
$68 \times 6 =$	$46 \times 3 =$	$34 \div 3 =$	$72 \div 6 =$
$55 \times 4 =$		$21 \div 7 =$	
THREE-DIGIT BY ONE-DIGIT MULTIPLICATION		THREE-DIGIT BY ONE-DIGIT DIVISION	
$301 \times 4 =$	$369 \times 6 =$	$357 \div 6 =$	$497 \div 7 =$
$648 \times 5 =$	$465 \times 8 =$	$508 \div 2 =$	$214 \div 5 =$
$135 \times 7 =$		$522 \div 4 =$	
TWO-DIGIT BY TWO-DIGIT MULTIPLICATION		FOUR-DIGIT BY TWO-DIGIT DIVISION	
$57 \times 17 =$	$32 \times 19 =$	$8,064 \div 12 =$	$1,425 \div 15 =$
$68 \times 52 =$	$83 \times 49 =$	$1,825 \div 25 =$	$2,280 \div 30 =$
$94 \times 76 =$		$3,350 \div 50 =$	



ACTIVITY F.3

Name: *“Share-Share-Compare”*

Type: *Routine*

About the Routine: This routine can also be a longer classroom activity, depending on how many problems students are asked to solve. Each person first solves problems independently, then has the chance to have a one-on-one with a peer to compare their thinking on the same problem.

Materials: Prepare a list of three to five problems that lend to being solved different ways.

- Directions:**
1. Students work independently to solve the full set of problems.
 2. Students write if they solved it mentally by naming the strategy they used. If they solved it by writing, they do not need to name a strategy.
 3. Once complete, everyone stands up with their page of worked problems.
 4. Students find a partner who is *not* at their table. When they find a partner, they high-five each other and begin Share-Share-Compare for the first problem:
 - Share: Partner 1 **shares** their method.
 - Share: Partner 2 **shares** their method.
 - Compare: Partners discuss how their methods **compared**:

If their methods are different, they compare the two, discussing which one worked the best or if both worked well.

If their methods are the same, they think of an alternative method and again discuss which method(s) worked well.

After the exchange, partners thank each other, raise their hands to indicate they are in search of a new partner, find another partner, and repeat the process of Share-Share-Compare for problem 2 (or any problem they haven't yet discussed).

Possible problem sets for this routine include the following:

TWO-DIGIT BY ONE-DIGIT MULTIPLICATION		TWO-DIGIT BY ONE-DIGIT DIVISION	
$78 \times 8 =$	$65 \times 9 =$	$26 \div 5 =$	$67 \div 4 =$
$59 \times 6 =$	$47 \times 3 =$	$34 \div 3 =$	$48 \div 6 =$
$37 \times 4 =$		$98 \div 7 =$	
THREE-DIGIT BY ONE-DIGIT MULTIPLICATION		THREE-DIGIT BY ONE-DIGIT DIVISION	
$482 \times 4 =$	$835 \times 7 =$	$287 \div 6 =$	$523 \div 4 =$
$292 \times 5 =$	$244 \times 6 =$	$327 \div 2 =$	$497 \div 7 =$
TWO-DIGIT BY TWO-DIGIT MULTIPLICATION		FOUR-DIGIT BY TWO-DIGIT DIVISION	
$61 \times 52 =$	$37 \times 19 =$	$4,450 \div 50 =$	$7,248 \div 12 =$
$19 \times 76 =$	$83 \times 49 =$	$1,125 \div 25 =$	$8,520 \div 40 =$



ACTIVITY F.4

Name: *Strategy Spin*

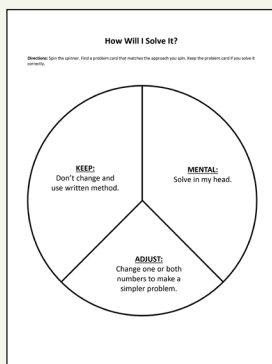
Type: *Game*

About the Game: This game can be played in many ways. You can have spinners that focus on metacognition (mental, adjust, keep; see metacognition visual), you can pick the strategies students know, or you can have the options “Standard Algorithm” and “Not Standard Algorithm.” There are many online spinner tools, such as Wheel Decide (<https://wheeldecide.com/>), which allow you to enter the categories you want and actually spin virtually. This game can be played with two to four players.

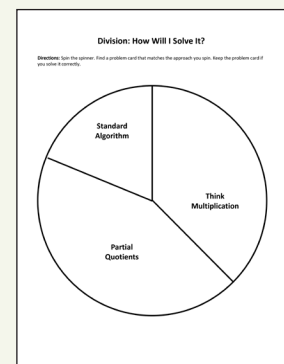
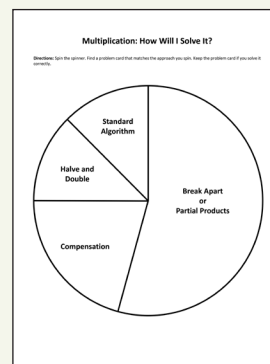
Materials: strategy spinner (one per group) and expression cards

- Directions:**
1. Players take turns spinning the strategy spinner.
 2. Once the spinner lands on a strategy, the player looks through the expression bank to find a problem they want to solve using that strategy.
 3. The player tells how to solve the selected problem using that strategy. Opponents check solutions using a Hundred Chart, number line, or calculator. If correct, the player claims that expression card.
 4. Repeat steps 2 and 3 three times for four players, four times for three players, and five times for two players.
 5. Together the group looks at the remaining expressions and labels which strategy they think is a good fit for each.

RESOURCE(S) FOR THIS ACTIVITY



59×8	31×8
35×6	68×6
52×4	55×4
27×9	17×9
67×3	46×3



Game cards and spinners can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY F.5

Name: *Just Right*

Type: *Game*

About the Game: This game board can be adapted to incorporate the strategies students have learned. Earlier in the year, it may have only two options; later in the year, it may have five options. Students may be tempted to use inefficient strategies in order to get four in a row. To counter this, require students to record the equations and the strategy they used.

Materials: expression cards (mixed strategies), *Just Right* game board, and two-sided counters

- Directions:**
1. Players take turns flipping cards.
 2. Player selects a “just right” strategy, and solves it thinking aloud or on paper.
 3. Opponents confirm accuracy with calculators.
 4. Having correctly solved the problem, the player places a marker on the strategy they used on the *Just Right* game board.
 5. The first player to place four markers in a row on the game board wins.

For example, Sydney draws 55×12 . She says, “My ‘just right’ strategy is Halve and Double. I think double 55 is 110 and half of 12 is 6. 110×6 is 660.” Her opponent, Annie, confirms that she is correct. Sydney covers her choice of a Halve and Double square.

RESOURCE(S) FOR THIS ACTIVITY

Just Right: Multiplication Board				
Directions: Flip over a problem card. Decide which strategy is “just right” for the problem. Place a marker on the strategy. Be the first to get four markers in a row (horizontally, vertically, or diagonally).				
Compensation	Break Apart	Halve and Double	Standard Algorithm	Partial Products
Partial Products	Halve and Double	Compensation	Break Apart	Compensation
Halve and Double	Standard Algorithm	Break Apart	Compensation	Partial Products
Partial Products	Halve and Double	Compensation	Standard Algorithm	Halve and Double
Compensation	Partial Products	Break Apart	Standard Algorithm	Partial Products

Just Right: Division Board				
Directions: Flip over a problem card. Decide which strategy is “just right” for the problem. Place a marker on the strategy. Be the first to get four markers in a row (horizontally, vertically, or diagonally).				
Standard Algorithm	Think Multiplication	Standard Algorithm	Partial Quotients	Standard Algorithm
Think Multiplication	Partial Quotients	Standard Algorithm	Think Multiplication	Partial Quotients
Partial Quotients	Think Multiplication	Standard Algorithm	Partial Quotients	Think Multiplication
Think Multiplication	Partial Quotients	Standard Algorithm	Think Multiplication	Partial Quotients
Standard Algorithm	Partial Quotients	Standard Algorithm	Think Multiplication	Standard Algorithm



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY F.6

Name: *Strategies*

Type: *Game*

About the Game: This game is an excellent opportunity for practice or assessment once a variety of strategies are learned. Students generate three to six examples of problems that lend to using that strategy, such as 29×7 (Compensation), 58×5 (Halve and Double), or 38×48 (Partial Products), relative to the size of factors to which they have been exposed. The directions describe this as a small group activity, but it can be modified to a center. An alternative to this game is to just focus on one strategy (e.g., Partial Products).

Materials: *Strategies* game card (one per student)

- Directions:**
1. Each player works independently to generate a problem for which they would use that strategy (alternatively, students can work with a partner to discuss problems that would fit in each strategy).
 2. After players have completed their *Strategies* game card (i.e., has one example expression in each strategy), place students in groups of three.
 3. On a player's turn, they ask one of the other group members to share the problem on their *Strategies* card for ____ strategy. If the player explains the problem using that strategy, they score 5 points. If not, the third player gets a chance to "steal" by explaining the problem using that strategy. If the third player cannot, the author of the problem must explain using that strategy. If they cannot, they lose 10 points.
 4. Play continues until all strategies have been solved, or students have played three rounds. The high score wins!
 5. To facilitate discussion after play, ask questions such as these: What do you notice about the problems in ____ strategy? When is this strategy useful?

RESOURCE(S) FOR THIS ACTIVITY

Strategies: Division	
Strategy	My Problem
Think Multiplication	
Think Multiplication	
Partial Quotients	
Partial Quotients	

Strategies: Multiplication	
Strategy	My Problem
Break Apart	
Halve and Double	
Compensation	
Partial Products	



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY F.7

Name: *Make the Product*

Type: *Game*

About the Game: As students encounter different two-digit factors, they will find expressions that lend to different strategies. That makes this a good game for observing what strategies students are commonly choosing to use (and whether they are using a variety of strategies). The game can be played as a center in which students play to get their personal best score.

Materials: two sets of digit cards or half a deck of playing cards (with face cards removed), *Make the Product* recording sheet

- Directions:**
1. Establish a target product. Note that the range for 2 two-digit factors is between 100 (10×10) and 9,801 (99×99). In this case, the product to make is 2,500.
 2. Each player is dealt six cards.
 3. Each player uses four of the six cards to make 2 two-digit factors with a product as close to 2,500 as possible.
 4. The player records their equation and shows how they found the product.
 5. Players record how far away they are from the target product (in this case, 2,500).
 6. After five rounds, each player adds the distance from 100 from each round for a score.
 7. The player with the lower score wins.

For a target of 2,500, Oscar pulls 5, 3, 4, 7, 2, and 1. He makes 43×57 and finds the product to be 2,451. He is 49 from the target.

RESOURCE(S) FOR THIS ACTIVITY

Make the Product _____

Directions: Deal six cards. Use four cards to make two factors with a product as close to the target as possible. Find the product and show how you multiplied. Record how far from the target you are for each round.

Round	My Cards	My Multiplication Problem	How Far From ____?
1			
2			



Digit cards and this recording sheet can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY F.8

Name: *A-MAZE-ing Race*

Type: *Game*

About the Game: This game is not about speed! Putting time pressure on students works against fluency, as the stress can block good reasoning. The purpose of this game is to practice selecting and using strategies that are a good fit for the numbers in the problem. It can be adapted for any multiplication or division problem sets. You can also use it as a center in which students highlight their path, record the problems, find the solutions, and show their strategy.

Materials: *A-MAZE-ing Race* game board (one per pair), two-sided counters, calculators

- Directions:**
1. Players both put their marker on Start.
 2. Players take turns selecting an unoccupied square that shares a border with their current position.
 3. Players talk aloud to say the answer and explain how they solved the equation. An optional sentence frame to use is, "The answer is __. I used __ strategy to solve it. I __, then I __."
 4. Opponents confirm accuracy with calculators. If correct, the player moves to the new square.
 5. Note that the dark lines cannot be crossed.
 6. The winner is the first to reach the finish.

RESOURCE(S) FOR THIS ACTIVITY

A-MAZE-ing Race

Directions: Choose an open space that touches the space you are in to move. Tell how you solve the problem. If correct, move to the new space. You cannot cross dark lines. Be the first to get to the finish.

START	$439 \div 6$	$39 \div 2$	$706 \div 4$	$81 \div 9$	$5,260 \div 25$
$907 \div 3$	$71 \div 9$	$1,739 \div 8$	$48 \div 3$	$990 \div 5$	$3,950 \div 7$
$89 \div 4$	$408 \div 4$	$47 \div 8$	$1,510 \div 8$	$2,538 \div 3$	$8,275 \div 9$
$65 \div 3$	$619 \div 2$	$2,671 \div 3$	$77 \div 7$	$991 \div 4$	$3,247 \div 8$
$248 \div 3$	$218 \div 9$	$734 \div 8$	$96 \div 7$	$54 \div 6$	$402 \div 7$
$2,735 \div 3$	$6,857 \div 8$	$2,015 \div 4$	$1,853 \div 9$	$4,495 \div 2$	FINISH

A-MAZE-ing Race

Directions: Choose an open space that touches the space you are in to move. Tell how you solve the problem. If correct, move to the new space. You cannot cross dark lines. Be the first to get to the finish.

START	19×3	319×6	7×24	161×9	6×49
376×9	179×2	39×3	4×23	9×26	186×3
97×5	481×7	4×58	51×8	285×3	817×4
6×33	519×3	8×29	7×177	99×5	697×2
48×5	8×17	73×8	196×7	274×6	99×9
88×4	698×3	27×5	813×8	999×8	FINISH

A-MAZE-ing Race

Directions: Choose an open space that touches the space you are in to move. Tell how you solve the problem. If correct, move to the new space. You cannot cross dark lines. Be the first to get to the finish.

START	18×46	48×27	26×29	39×46	56×14
19×19	26×14	97×58	39×27	99×99	26×29
43×34	22×39	52×24	88×24	33×38	24×18
49×51	48×49	26×22	99×98	71×29	56×37
34×53	42×17	53×8	66×79	74×67	98×99
88×48	26×52	27×53	43×18	97×97	FINISH



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.



ACTIVITY F.9

Name: **Strategy Problem Sort**

Type: **Center**

About the Center: Just because a problem can be solved with a given strategy does not mean it is a good fit. It is a good fit when it is the most appropriate option from a student's set of known strategies. (See the Part 1 discussion of appropriate strategies.) You can extend the activity by having students go through the "Does Not Fit the Strategy" category and identify which strategy would work well for that problem.

Materials: Strategy Problem Sort cards or a set of 12–20 mixed problems (some that fit the intended strategy and some that do not), Strategy Problem Sort placemat (optional)

- Directions:**
1. Students flip over a problem card.
 2. Students determine if the problem fits the strategy or doesn't fit the strategy.
 3. Students then solve the problem.

To create assessment artifacts, you can take a picture of a student's completed sort. Or, you can ask students to provide written responses to questions like these:

I placed ____ in the "Fits the Strategy" side because . . .

I placed ____ in the "Does Not Fit the Strategy" side because . . .

RESOURCE(S) FOR THIS ACTIVITY

Strategy Sort Placemat	
Strategy: _____	
Fits the Strategy	Doesn't Fit the Strategy

39×7	44×74
51×62	899×3
298×4	$9 \times 4,177$
$3,299 \times 2$	95×8
61×9	374×6



These resources can be downloaded at resources.corwin.com/FOF/multiplydividewholenumber.

Appendix

Tables of Activities

Figuring out fluency is a journey. Fluency with whole number multiplication and division is absolutely essential for life and for future mathematics. Each and every child must have access and ample opportunities to develop their understanding and use of reasoning strategies. It is critical to remember these points:

- Fluency needs to be a daily part of mathematics instruction.
- There are no shortcuts or quick fixes to developing fluency.
- Fluency requires instruction *and* ongoing reinforcement.
- Different students require different types and quantities of experiences to develop fluency.
- While strategy choice is about the individual student, every student must learn and practice significant strategies so that they *can* choose to use them.
- Fluency practice must not be stressful. Stress complicates thinking.
- Fluency is more than accuracy; you must assess the other components.

This book is packed with activities for instruction, practice, and assessment to support the work that you do and to supplement the resources you use. The following pages provide a listing of all of the activities in this book. These tables can help you achieve the following:

- Jump between strategies, as you may not teach them sequentially.
- Locate prompts for teaching each strategy.
- Identify a specific type of activity to incorporate into your fluency instruction.
- Identify activities for specific strategies that you need to reteach, reinforce, or assess.
- Take notes about revisiting an activity later in the year.
- Take notes about modifying an activity for use with another strategy.
- Take notes about how you might leverage the activity in future years.
- Identify an activity that is particularly useful for assessment.

MODULE 1: BREAK APART TO MULTIPLY ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
1.1	25	T	Multiples of 10	
1.2	26	T	Color Tile Arrays	
1.3	27	T	Breaking Apart Basic Facts	
1.4	27	T	Two Cuts	
1.5	28	T	The Better Break	
1.6	29	T	Breaking Apart With Multiplication	
1.7	30	T	Prompts for Teaching Break Apart to Multiply	
1.8	33	R	“Complex Number Strings”	
1.9	34	R	“The Breaks” (breaking apart factors)	
1.10	35	R	“Strategize Your Break Apart”	
1.11	36	G	<i>Factor Race</i>	
1.12	37	G	<i>Mult-zee</i>	
1.13	38	G	<i>Break Apart Bingo</i>	
1.14	39	G	<i>Break Apart to Multiply Math Libs</i>	
1.15	40	C	Same But Different	
1.16	41	C	Backward Breaks	
1.17	42	C	Break Apart With an Area Model Representation	
1.18	43	C	Which and How?	

T (Teaching) • R (Routine) • G (Game) • C (Center/Independent)

MODULE 2: HALVE AND DOUBLE ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
2.1	46	T	Halving and Rearranging Rectangles	
2.2	47	T	Cuisenaire Rod Proofs	
2.3	48	T	Representing Halve and Double With Graphic Organizers	
2.4	49	T	Nickels and Dimes	
2.5	50	T	Prompts for Teaching Halve and Double	
2.6	53	R	"A String of Halves" (practice halving)	
2.7	54	R	"The Same As" (restating a problem with Halve and Double)	
2.8	55	G	<i>The Splits</i>	
2.9	56	G	<i>Get 10</i> (when to use the strategy)	
2.10	57	G	<i>Halve and Double Flips</i>	
2.11	58	G	<i>The First to Seven</i>	
2.12	59	C	Fives and Tens	
2.13	60	C	Write It as Halve and Double	

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MODULE 3: COMPENSATION ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
3.1	64	T	Show Me	
3.2	65	T	Compensated Trains (with linking cubes)	
3.3	66	T	True or False Compensation Statements	
3.4	66	T	Number Strings (Compensation focus)	
3.5	67	T	Prompts for Teaching Compensation	
3.6	70	R	"A Little More or A Little Less"	
3.7	71	R	"Or You Could" (restating with Compensation)	
3.8	72	G	<i>Compensation Points</i>	
3.9	73	G	<i>Give Some to Get Some</i>	
3.10	74	G	<i>Compensation Concentration</i>	
3.11	75	G	<i>Estimation Tic-Tac-Toe</i>	
3.12	76	C	Creating Compensations	
3.13	77	C	Prove It	
3.14	78	C	Use This for Solving	

T (Teaching) • R (Routine) • G (Game) • C (Center/Independent)

MODULE 4: PARTIAL PRODUCTS ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
4.1	83	T	Multiples Matrix	
4.2	84	T	Base-10 Partial	
4.3	84	T	Partial Products With Area Models	
4.4	85	T	Is It the Same? (comparing different ways to break apart)	
4.5	86	T	One or Both? (which factors to decompose)	
4.6	87	T	Same Digits, Different Product	
4.7	88	T	Prompts for Teaching Partial Products	
4.8	91	R	“Why Not?”	
4.9	92	R	“About or Between” (estimation)	
4.10	93	G	<i>Make It Close</i>	
4.11	94	G	<i>Connect Four Partial</i>	
4.12	95	G	<i>Target 2,500</i>	
4.13	96	C	What’s the Problem	
4.14	97	C	Pick Your Partial	
4.15	98	C	Guess and Check—The Largest Product	
4.16	99	C	Bag of Blocks (partials with base-10 blocks)	
4.17	100	C	One or Both?	

T (Teaching) • R (Routine) • G (Game) • C (Center/Independent)

MODULE 5: THINK MULTIPLICATION ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
5.1	104	T	Is It Still True?	
5.2	105	T	The Direct Connect: Three Strings	
5.3	106	T	Recognizing Products and Decomposing Dividends	
5.4	107	T	What's Missing? (with triangle cards)	
5.5	107	T	Prompts for Teaching Think Multiplication	
5.6	110	R	"If I Know . . ."	
5.7	111	R	"The Count Up" (counting by multiples to find dividends)	
5.8	112	R	"Breaking Dividends"	
5.9	113	G	<i>The Take</i>	
5.10	114	G	<i>Paper Clip Connections</i>	
5.11	115	G	<i>Covering the Breaks</i>	
5.12	116	G	<i>Area Model Duel</i>	
5.13	117	C	Make the Quotient	
5.14	118	C	Rewrite It and Solve It	
5.15	119	C	Triangle Cards	
5.16	120	C	Finding the Breaks	

T (Teaching) • R (Routine) • G (Game) • C (Center/Independent)

MODULE 6: PARTIAL QUOTIENTS ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
6.1	124	T	Base-10 Quotients	
6.2	125	T	Is It the Same? (comparing different partial quotients)	
6.3	125	T	Another Way (representing partial quotients)	
6.4	126	T	Comparing Partial	
6.5	126	T	Prompts for Teaching Partial Quotients	
6.6	130	R	“How Many Are In?”	
6.7	131	R	“Could It Be?”	
6.8	132	G	<i>Quotient Hunt</i>	
6.9	134	G	<i>End of the Lane</i>	
6.10	135	G	<i>Quotient Connect</i>	
6.11	136	G	<i>Remainder Tally</i>	
6.12	137	G	<i>Division Bump</i>	
6.13	138	G	<i>Tic-Tac-Take</i>	
6.14	139	C	How Many Are In?	
6.15	140	C	Estimate and Divide	
6.16	141	C	The Largest Quotient, The Smallest Quotient	
6.17	142	C	Changing Dividends Make for Changing Quotients	

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MODULE 7: STANDARD ALGORITHM ACTIVITIES				
NO.	PAGE	TYPE	NAME	NOTES
7.1	147	T	Connecting Partial Products and the Algorithm	
7.2	148	T	Connecting Partial Quotients and the Algorithm	
7.3	148	T	Find the Error	
7.4	149	T	To Algorithm or Not to Algorithm	
7.5	150	T	Prompts for Teaching Standard Algorithms	
7.6	153	R	“Over/Under” (estimation)	
7.7	154	R	“That One” (when to use the algorithm)	
7.8	155	G	<i>Call Your Shot</i>	
7.9	156	G	<i>A List of Ten</i>	
7.10	157	G	<i>For Keeps</i>	
7.11	158	G	<i>The V Cover Up</i>	
7.12	159	C	Missing Numbers With Multiplication Algorithms	
7.13	160	C	Missing Digits With Division Algorithms	
7.14	161	C	Partials to Algorithms	
7.15	162	C	Making Connections	
7.16	163	C	Correct the Error	

T (Teaching) • R (Routine) • G (Game) • C (Center/Independent)

PART 3: CHOOSING AND USING STRATEGIES				
NO.	PAGE	TYPE	NAME	NOTES
F.1	169	R	"Strategize First Steps"	
F.2	171	R	"M-A-K-E a Decision"	
F.3	172	R	"Share–Share–Compare"	
F.4	173	G	<i>Strategy Spin</i>	
F.5	174	G	<i>Just Right</i>	
F.6	175	G	<i>Strategies</i>	
F.7	176	G/C	<i>Make the Product</i>	
F.8	177	G/C	<i>A-MAZE-ing Race</i>	
F.9	178	C	Strategy Problem Sort	

T (Teaching) • R (Routine) • G (Game) • C (Center/Independent)

References

- Baroody, A. J., & Dowker, A. (Eds.). (2003). *Studies in mathematical thinking and learning. The development of arithmetic concepts and skills: Constructing adaptive expertise*. Lawrence Erlbaum Associates.
- Baroody, A. J., Purpura, D. J., Eiland, M. D., Reid, E. E., & Paliwal, V. (2016). Does fostering reasoning strategies for relatively difficult basic combinations promote transfer by K–3 students? *Journal of Educational Psychology*, 108(4), 576–591. <https://psycnet.apa.org/doi/10.1037/edu0000067>
- Bay-Williams, J., & Kling, G. (2019). *Math fact fluency: 60+ games and assessment tools to support learning and retention*. ASCD.
- Boaler, J. (2015a). *What's math got to do with it? How teachers and parents can transform mathematics learning and inspire success*. Penguin Books.
- Boaler, J. (2015b). *Memorizers are the lowest achievers and other Common Core math surprises*. <https://hechingerreport.org/memorizers-are-the-lowest-achievers-and-other-common-core-math-surprises/>
- Brendefur, J., Strother, S., Thiede, K., & Appleton, S. (2015). Developing multiplication fact fluency. *Advances in Social Sciences Research Journal*, 2(8), 142–154. <https://doi.org/10.14738/assrj.28.1396>
- Cheind, J., & Schneider, W. (2012). The brain's learning and control architecture. *Current Directions in Psychological Science*, 21(2), 78–84. <https://www.jstor.org/stable/23213097?seq=1>
- “Explicit.” (2021). *Merriam-Webster.com*. <https://www.merriam-webster.com/dictionary/explicit>
- Fuson, K. C., & Beckmann, S. (2012–2013, Fall/Winter). Standard algorithms in the Common Core State Standards. *NCSM Journal*, 14–30.
- Franke, M. L., Kazemi, E. & Turrou, A. C., (2018). *Choral counting and counting collections: Transforming the PreK–5 math classroom*. Stenhouse Publishers.
- Jameson, M. M. (2013). Contextual factors related to math anxiety in second-grade children. *Journal of Experimental Education*, 82(4), 518–536. <https://doi.org/10.1080/00220973.2013.813367>
- Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, 45(3), 850–867. <https://doi.org/10.1037/a0014939>
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academy Press. <https://doi.org/10.17226/9822>
- Locuniak, M. N., & Jordan, N. C. (2008). Using kindergarten number sense to predict calculation fluency in second grade. *Journal of Learning Disabilities*, 41(5), 451–459. <https://doi.org/10.1177/0022219408321126>
- National Center for Education Statistics (NCES). NAEP Report Card: 2019 NAEP Mathematics Assessment. <https://www.nationsreportcard.gov/mathematics/nation/achievement/?grade=4>
- National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.
- O’Connell, S., & SanGiovanni, J. (2014). *Mastering the basic math facts in multiplication and division: Strategies, activities, and interventions to move students beyond memorization*. Heinemann.
- Parrish, S. (2014). *Number Talks: Helping children build mental math and computation strategies, Grades K–5*. MathSolutions.
- Purpura, D. J., Baroody, A. J., Eiland, M. D., & Reid, E. E. (2016). Fostering first graders’ reasoning strategies with basic sums: The value of guided instruction. *The Elementary School Journal*, 117(1), 72–100. <https://doi.org/10.1086/687809>

- Ramirez, G., Shaw, S. T., & Maloney, E. A. (2018). Math anxiety: Past research, promising interventions, and a new interpretation framework. *Educational Psychologist*, 53(3), 145–164. <https://doi.org/10.1080/00461520.2018.1447384>
- SanGiovanni, J. J. (2019). *Daily routines to jump-start math class: Elementary school*. Corwin.
- SanGiovanni, J., Katt, S., & Dykema, K. (2020). *Productive math struggle: A six-point action plan for fostering perseverance*. Corwin.
- Star, J. R. (2005). Reconceptualizing conceptual knowledge. *Journal for Research in Mathematics Education*, 36(5), 404–411. <https://doi.org/10.2307/30034943>

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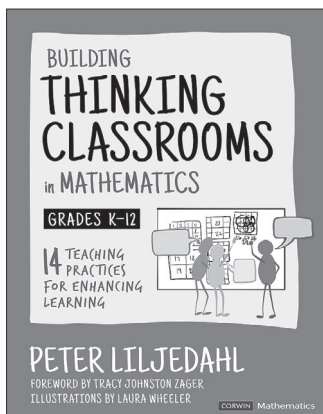
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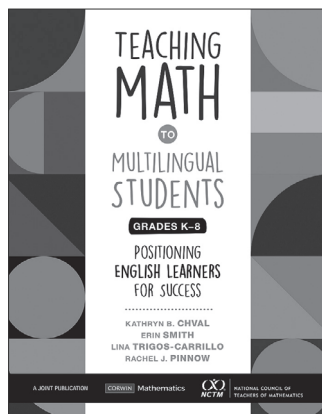
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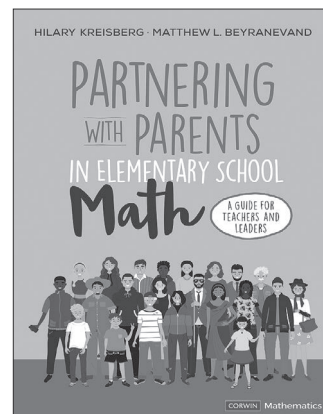
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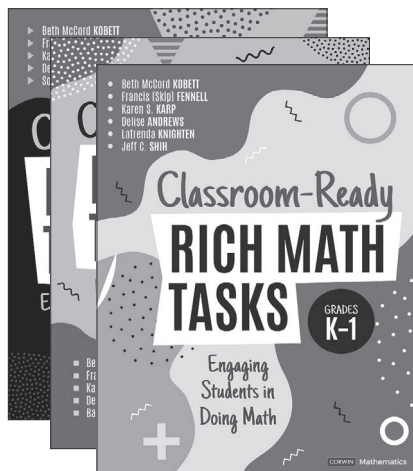


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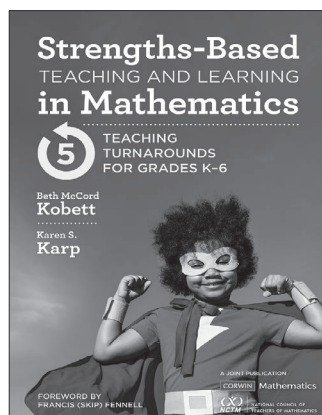


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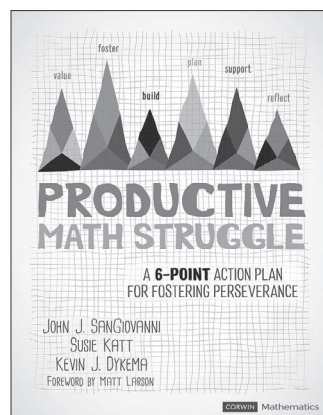
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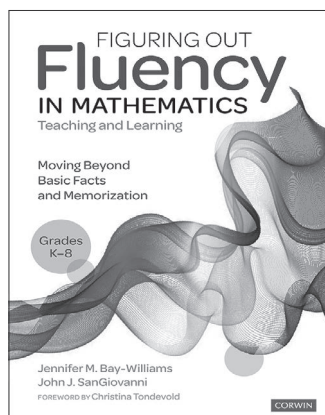
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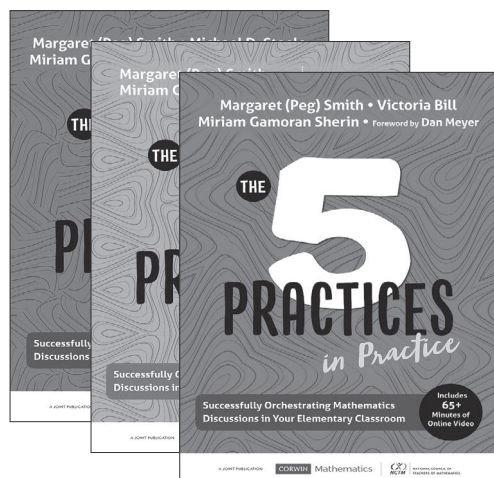
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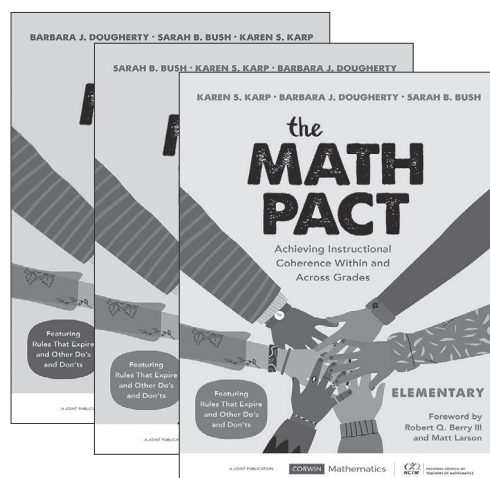
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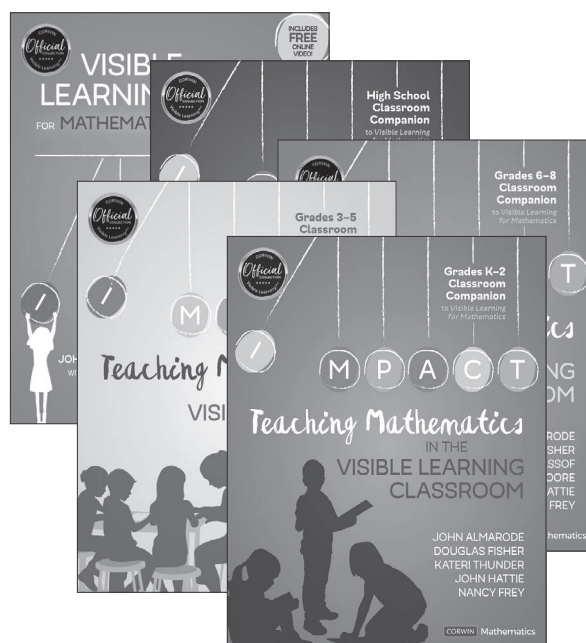
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