



Foundations of Numerical Reasoning in Grades 3-5

Module 3: Building Fluency Through 100 (Mult & Div)

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6 Week Module Breakdown

Week 1: Whole Number and Place Value

Week 2: Building Fluency through 1000 (Add & Sub)

Week 3: Building Fluency through 100 (Mult & Div)

Week 4: Building Fluency through 1000 (Mult & Div)

Week 5: Extending Place Value and Operations (Decimals)

Week 6: Number Statistical Reasoning

Module Breakdown

Part 1: Launch the Concept

Part 2: Connect and Extend Thinking

Week 3: Building Fluency



3-Act Task

The Peach Picker



Act 1

Pick Your Own
**PEACHES
& BERRIES**



Notice?

Pause
II

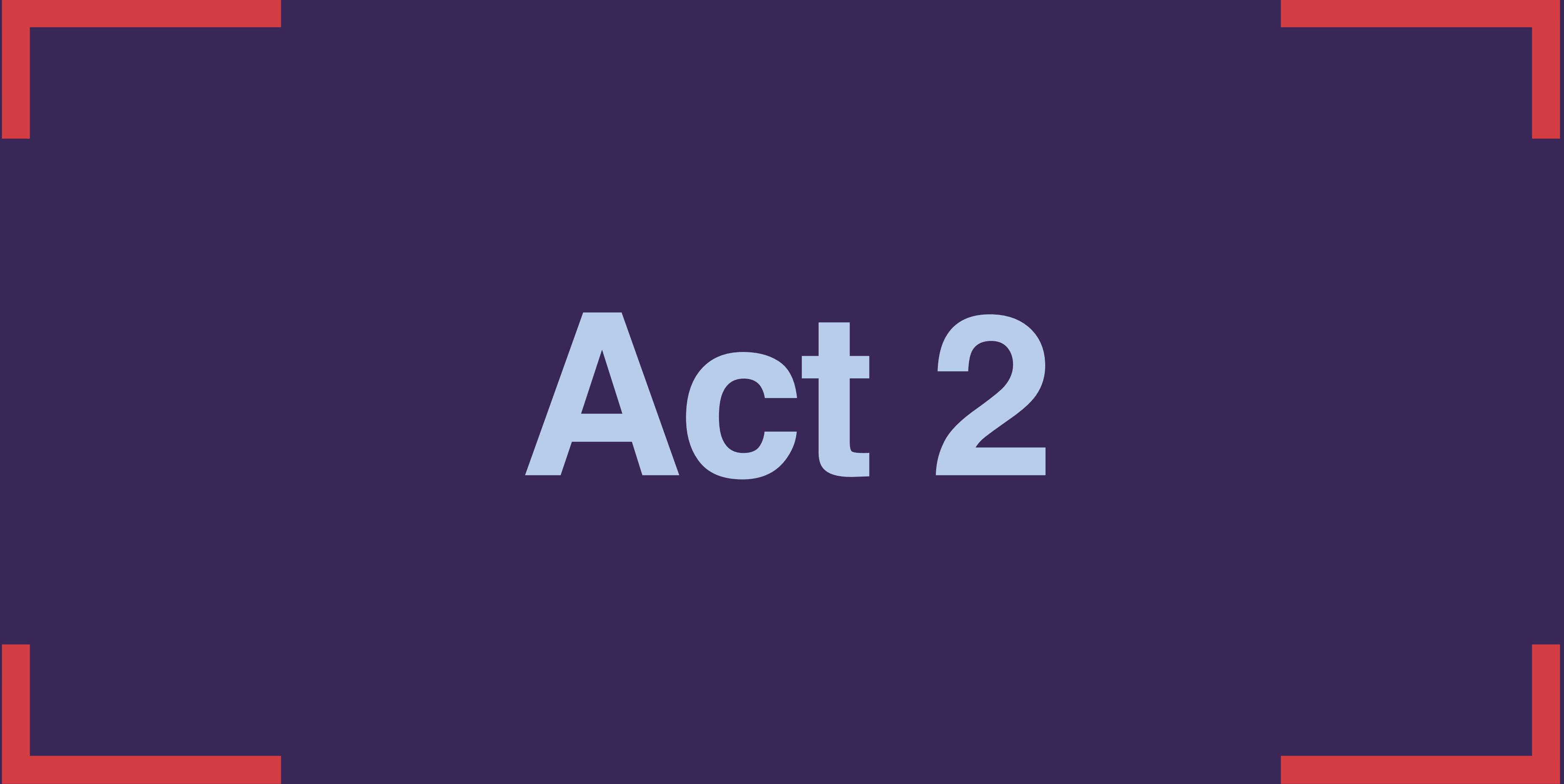
Wonder?



How many peaches in the basket?

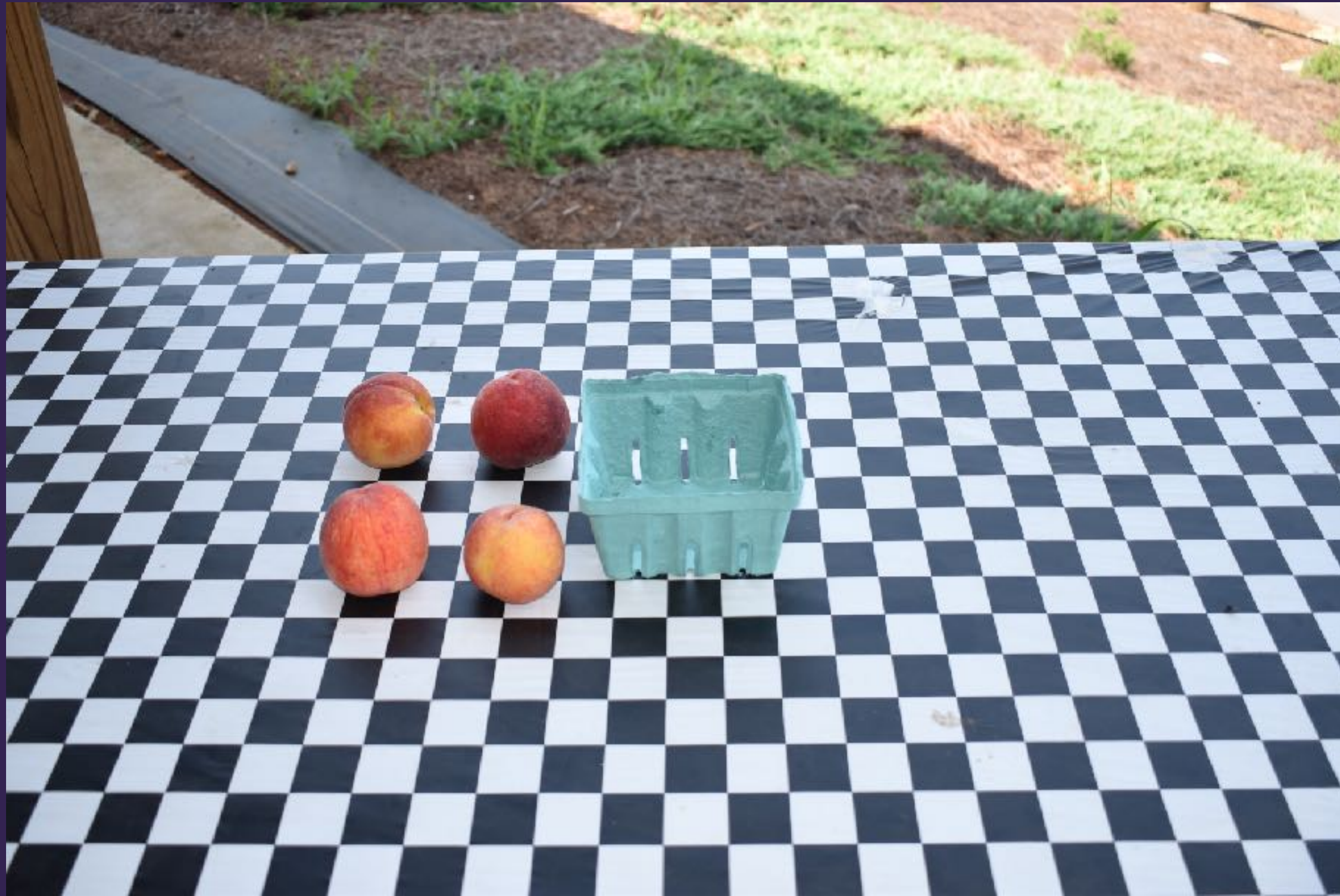
Pause
II





Act 2

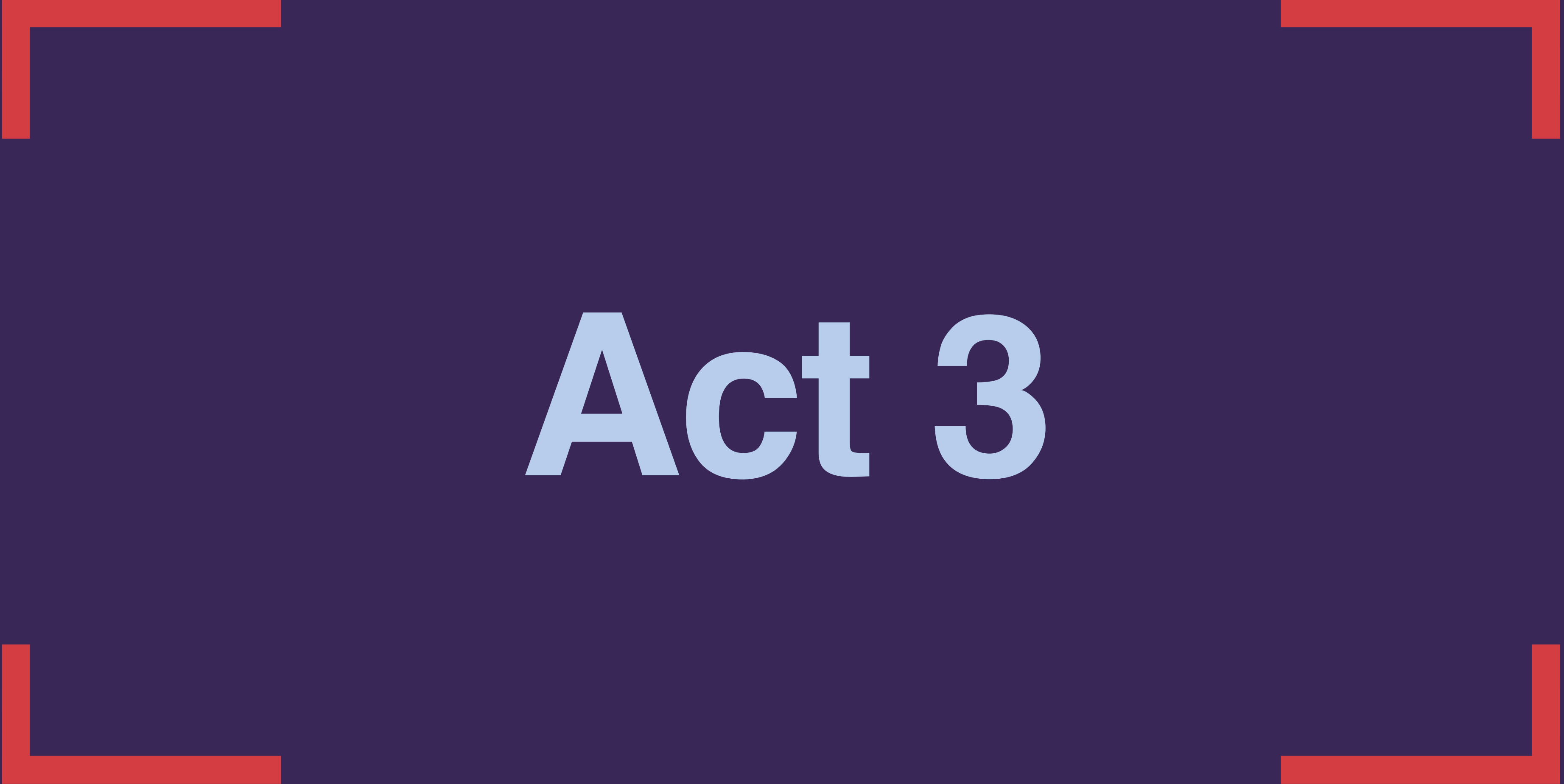
What information do you need?



What information do you need?

Pause
II





Act 3

Pick Your Own
**PEACHES
& BERRIES**





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Orchestrating Discussions

Five practices constitute a model for effectively using student responses in whole-class discussions that can potentially make teaching with high-level tasks more manageable for teachers.

Margaret S. Smith, Elizabeth K. Hughes, Randi A. Engle, and Mary Kay Stein



Margaret S. Smith, pegs@pitt.edu, is an associate professor of mathematics education at the University of Pittsburgh. Over the past decade, she has been developing research-based materials for use in the professional development of mathematics teachers and studying what teachers learn from the professional development in which they engage. Elizabeth K. Hughes, elizabeth.hughes@uni.edu, recently finished her doctorate in mathematics education at the University of Pittsburgh. Her areas of interest include preservice secondary mathematics teacher education and the use of practice-based materials in developing teachers' understanding of what it means to teach and learn mathematics. Randi A. Engle, raengle@berkeley.edu, is an assistant professor of mathematics education and the social context of learning at the University of California Berkeley. She is interested in developing practical theories for how mathematics teachers can create discussion-based learning environments that promote strong student engagement, learning, and transfer. Mary Kay Stein, mstein@pitt.edu, is a professor of learning sciences and policy and the director of the Learning Policy Center at the University of Pittsburgh. Her research focuses on instructional practice and the organizational and policy conditions that shape it.

Discussions that focus on cognitively challenging mathematical tasks, namely, those that promote thinking, reasoning, and problem solving, are a primary mechanism for promoting conceptual understanding of mathematics (Hatano and Inagaki 1991; Michaels, O'Connor, and Resnick forthcoming). Such discussions give students opportunities to share ideas and clarify understandings, develop convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives (NCTM 2000).

Although discussions about high-level tasks provide important

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The 5 Practices

1. **Anticipating** student responses to challenging mathematical tasks;
2. **Monitoring** students' work on and engagement with the tasks;
3. **Selecting** particular students to present their mathematical work;
4. **Sequencing** the student responses that will be displayed in a specific order and;
5. **Connecting** different students' responses and connecting the responses to key mathematical ideas.

Learning Target:

Questions and Look-Fors:

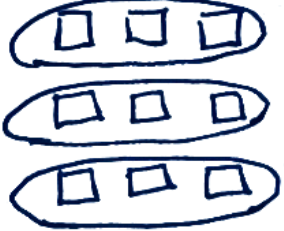
Strategy	Who and What	Order
		

Notes:

Anticipate



Pause
II

Learning Target: move thinking from additive to multiplicative		
Questions and Look-Fors: Students that <u>DON'T</u> use counting strategies		
Strategy	Who and What	Order
Doubling $4+4+4+4+4+4+4+4$ $\begin{array}{cccc} \underbrace{4+4}_8 & \underbrace{4+4}_8 & \underbrace{4+4}_8 & \underbrace{4+4}_8 \\ & \underbrace{8+8}_{16} & & \underbrace{8+8}_{16} \\ & & 16 & 16 \\ & & & 32+4=36 \end{array}$		
Draw all, count all $1, 2, 3 \dots 34, 35, 36$ Skip count $4, 8, 12 \dots 32, 36$		
Just Knew it $9 \times 4 = 36$ $(10 \times 4) - 4 = 36$		
Distributive Property $(5 \times 4) + (4 \times 4)$ $20 + 16 = 36$ 		
Associative Property $(4 \times 3) \times 3$ $4 \times (3 \times 3)$ 		
Notes: Extension: How might you share the peaches in baskets evenly but <u>NOT</u> in groups of 4? 2×18 3×12 6×6 18×2 12×3 4×9		

Anticipating → Monitoring → Selecting → Sequencing → Connecting

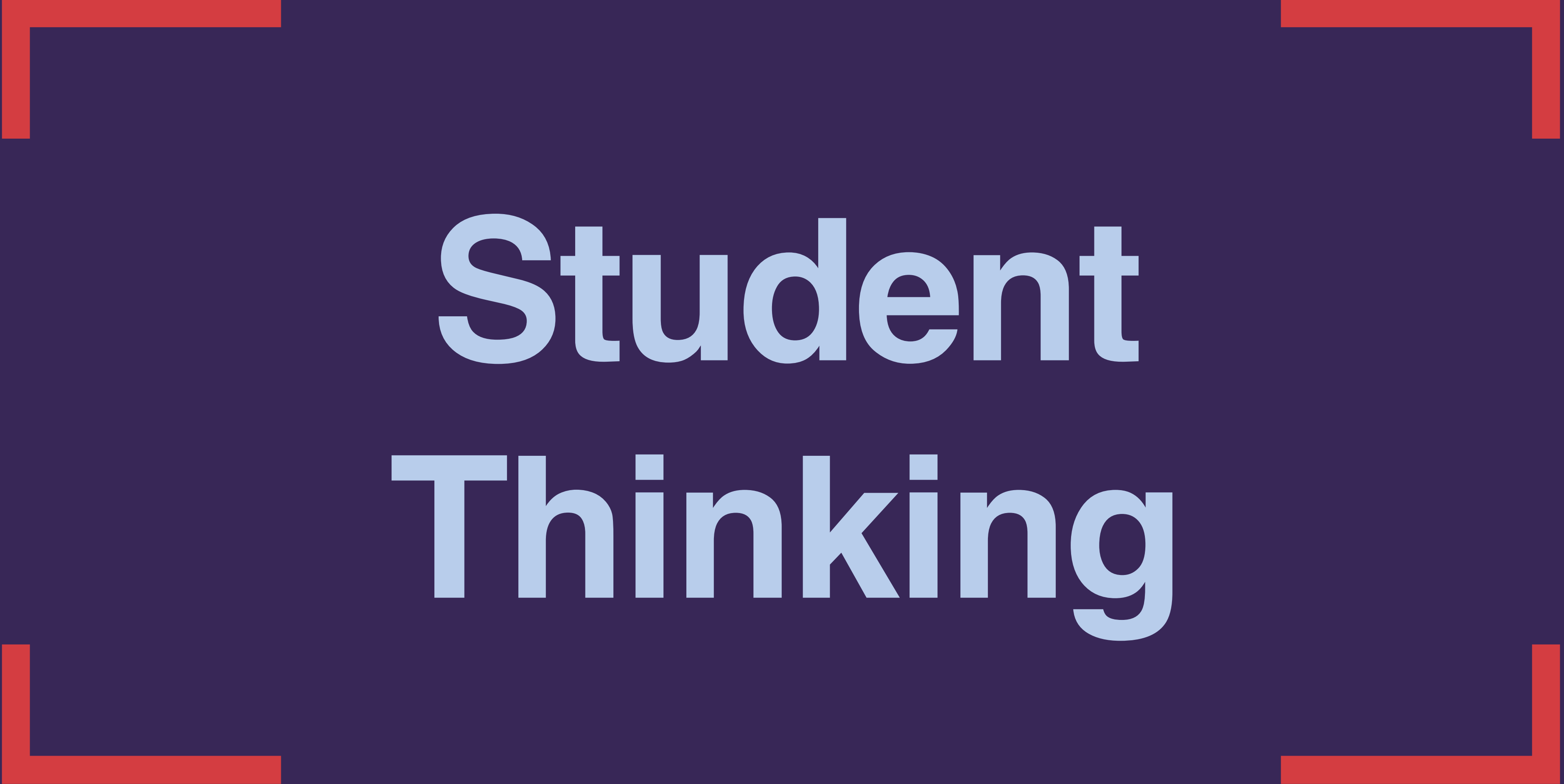
Anticipate





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Student Thinking

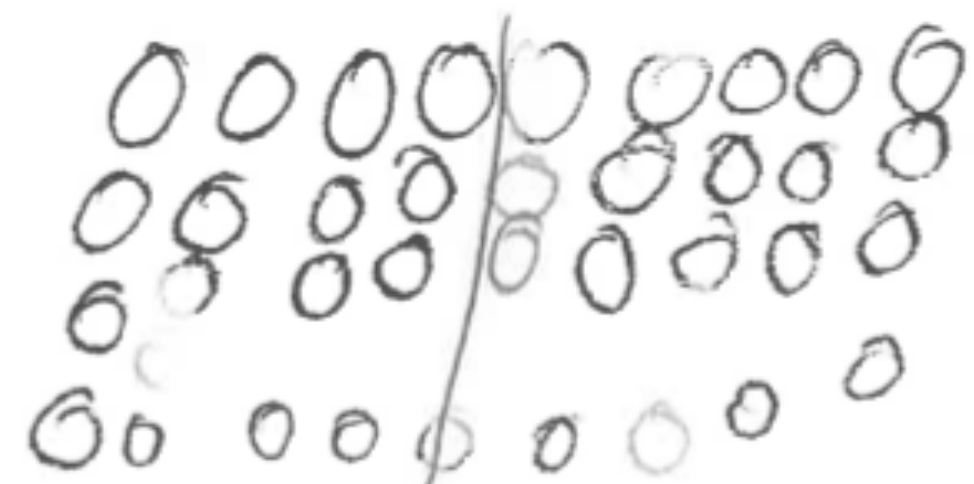
A-RUN

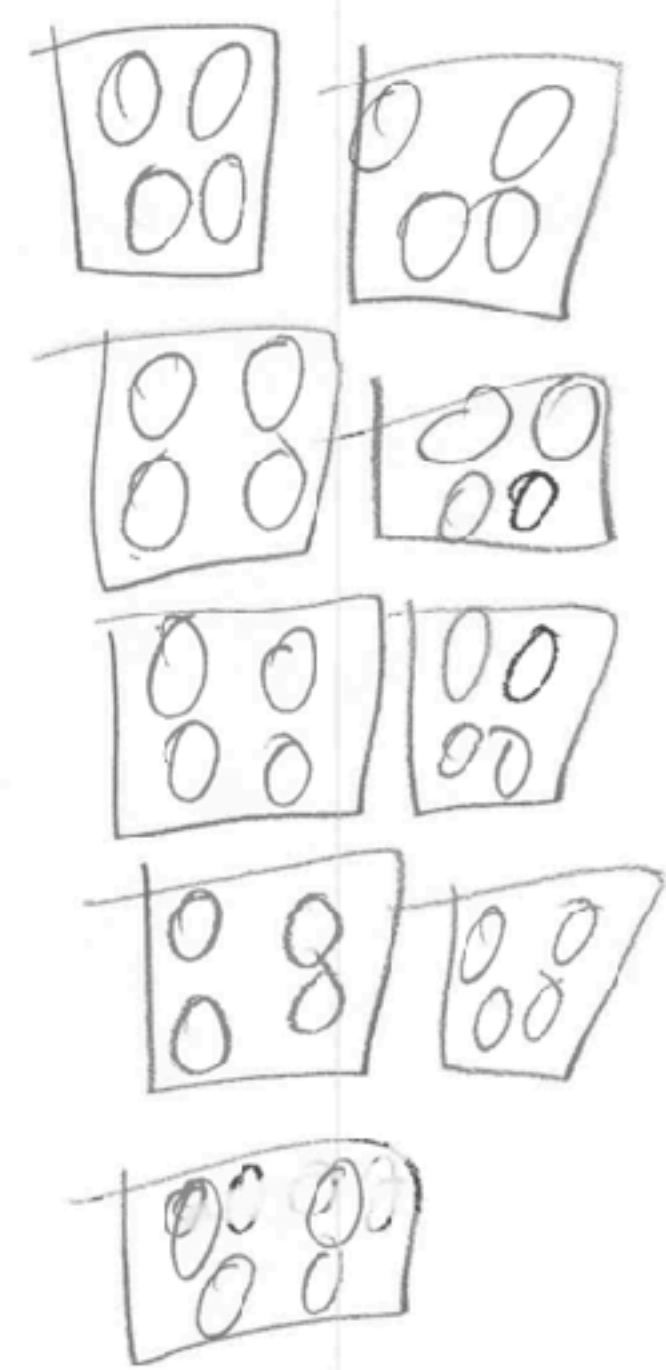
$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

case ☐ their were 9 peaches
and 9 BUCKETS

S1

S2


$$4 \times 4 = 16 \quad 5 \times 4 = 20$$
$$16 + 20 = 36$$



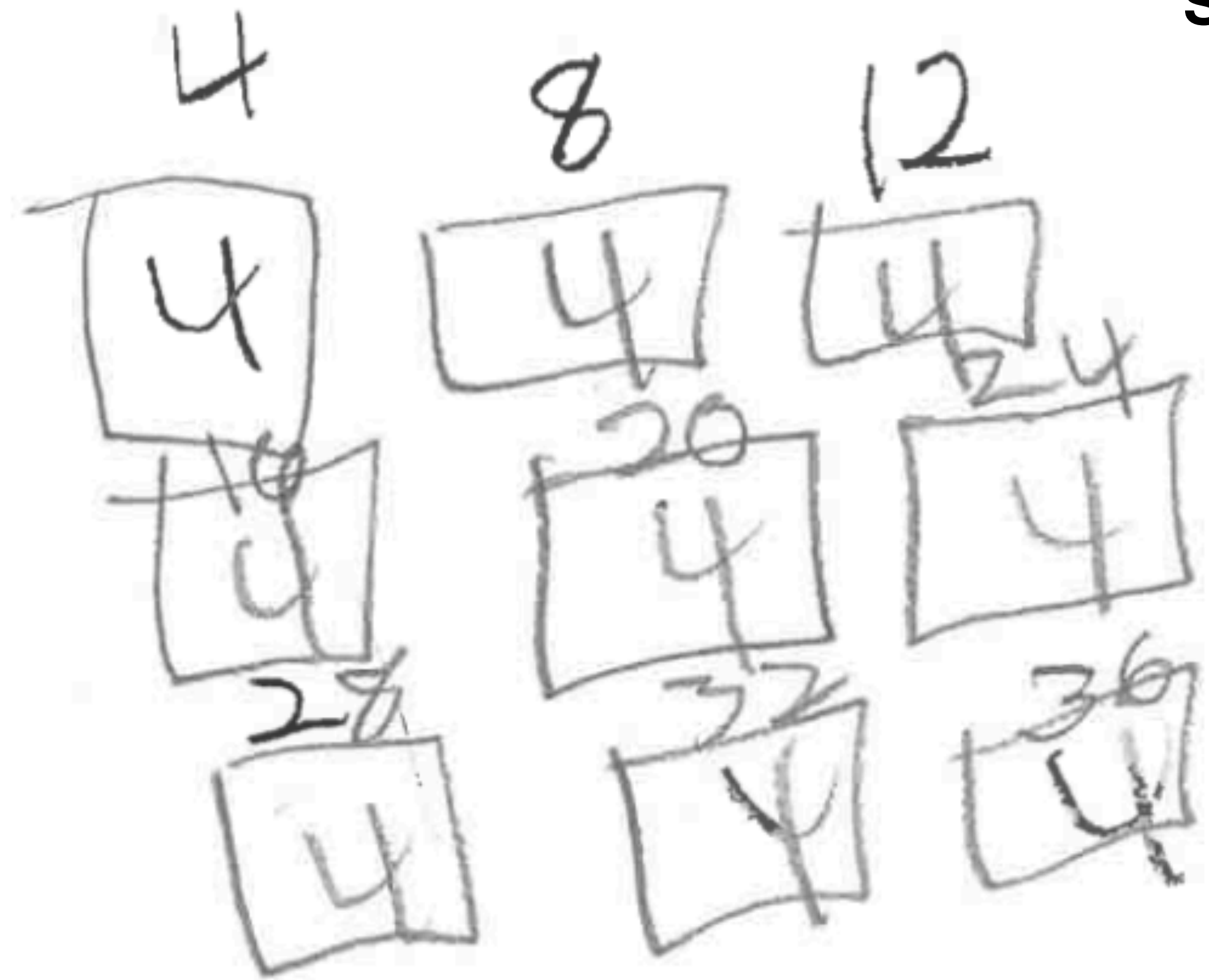
4 8 12 16 20 24 28 32 36

36 Peaches

$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

S3

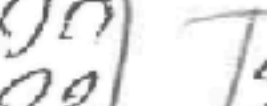
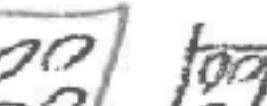
S4



$$\begin{array}{r} 6 \overline{) 36} \\ \underline{36} \\ 0 \end{array}$$

$$\begin{array}{r} 79 \\ + 36 \\ \hline 115 \end{array}$$

$$\begin{array}{r} 12 \\ \times 3 \\ \hline 36 \end{array}$$

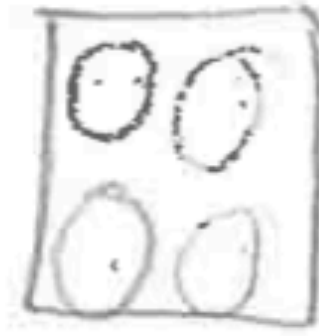
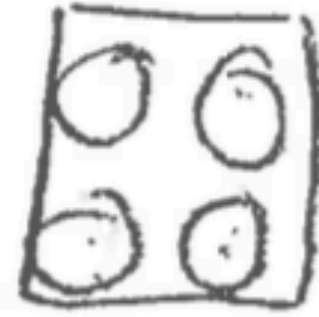
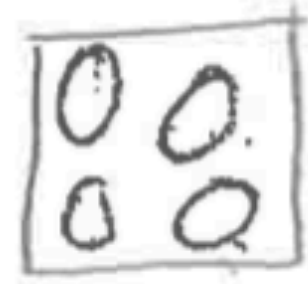

S5

4 | 8 12 + 12 + 12 = 36

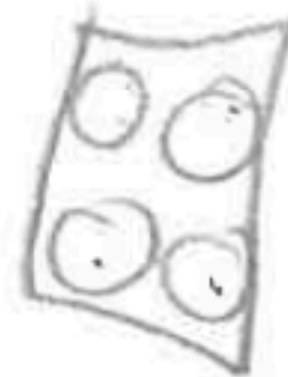
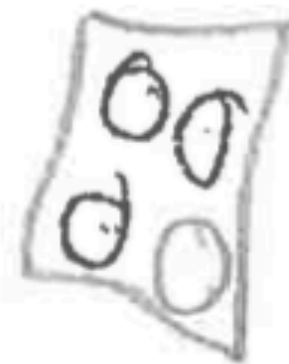
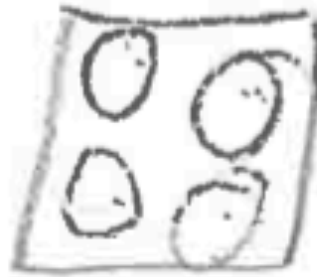
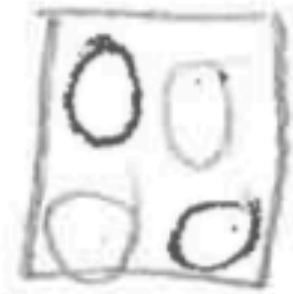
$$12 + 3 = 30$$

S 6

CYNTHIA



36



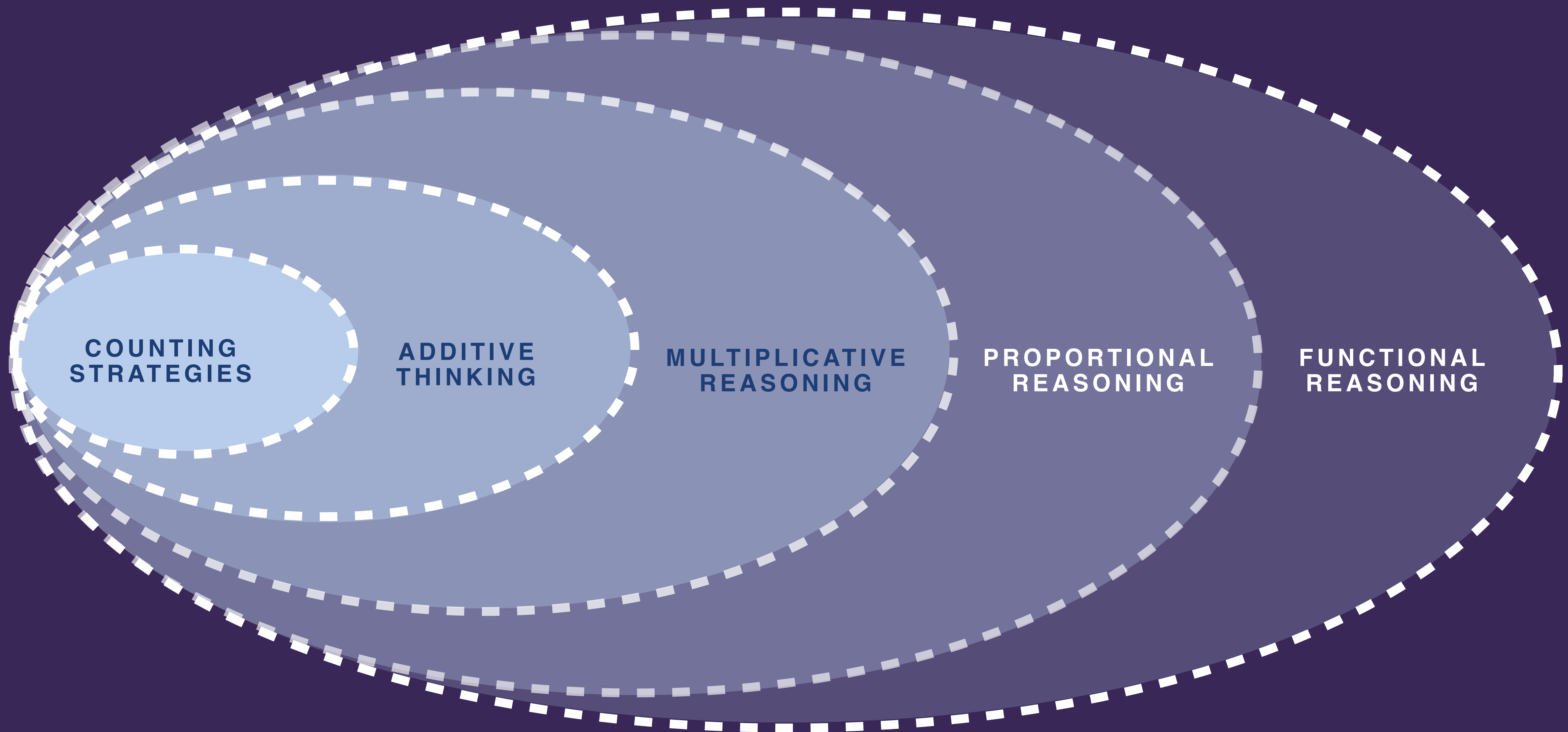
S7



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DEVELOPMENT OF MATHEMATICAL REASONING



$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

case ☐ their were 9 peaches
and 9 ☐ BUCKITS

S 2

Hand-drawn diagram showing two groups of circles. The first group is a 4x4 grid of 16 circles. The second group is a 5x4 grid of 20 circles. Below the grids, the calculations are written: $4 \times 4 = 16$, $5 \times 4 = 20$, and $16 + 20 = 36$.

S 3

A handwritten multiplication table for the number 4. The table is organized into three columns based on the multiplier: 4, 8, and 12. Each column contains three boxes, each containing the number 4. To the left of each box is a handwritten multiplier. The multipliers are 1, 2, and 3 for the first column; 5, 10, and 20 for the second column; and 12, 24, and 36 for the third column. The boxes are arranged in a grid-like fashion, with the multipliers written to the left of each box.

	4	8	12
1	4	4	4
2	4	4	4
3	4	4	4

S5

Pause

 $CYN + h\nu$

Hand-drawn illustration of a 3x3 grid of squares, each containing four dots in different arrangements. To the right is a large circle containing the number 36. Below the grid is the text "S7".

S 7

S 6

GFletchy

A-RON

$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

case = their were 9 peaches
and 9 Buckitts

Skip Counting **S1**

S2

$4 \times 4 = 16$ $5 \times 4 = 20$

Distributive Property

$$16 + 20 = 36$$

$48 | 262024283236$

Skip Counting

36 peaches

$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

S3

S4

Skip Counting

$6 | 9$
 $7 | 36$

79
 $+36$
 45

12
 $\times 3$
 36

Associative Property **S5**

$4 | 812$ $12 \times 3 = 36$

Associative Property

S6

CYNTHIA

Draw All Count All

S7



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A-RON

$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

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S1

S2

$$4 \times 4 = 16 \quad 5 \times 4 = 20$$

$$16 + 20 = 36$$

48 | 26 20 24 28 32 36

36 Peaches

$$4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

S3

S4

S5

6 | 9

7 | 36

79

+ 36

45

$$9 + 12 + 15 = 36$$

12

x 3

36

S6

4 | 8 12 + 12 + 12 = 36

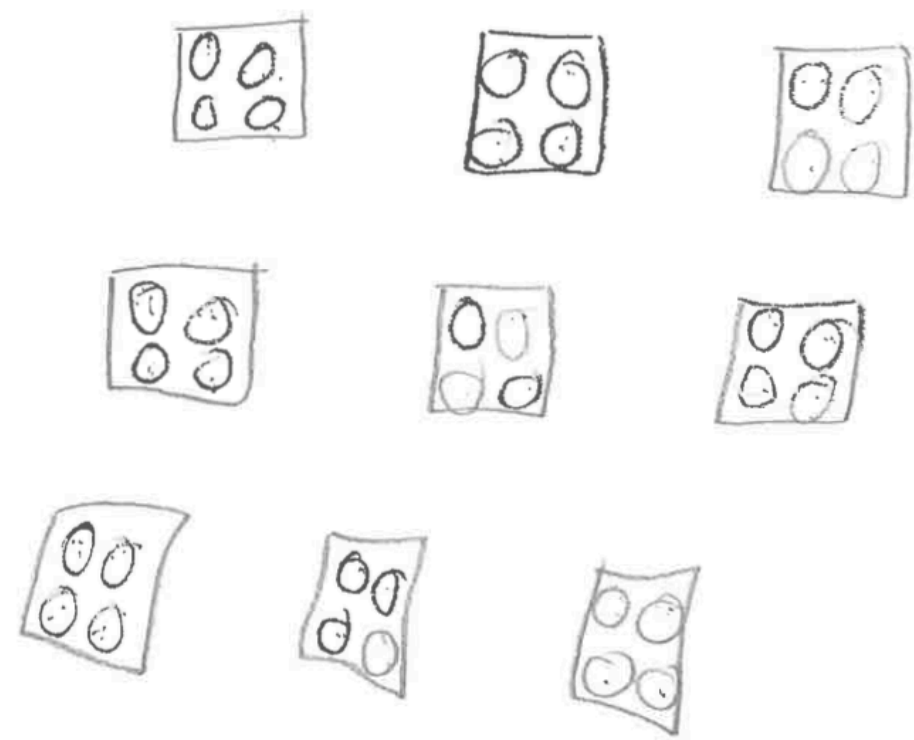
12 + 3 = 36

CYNTHIA

36

S7

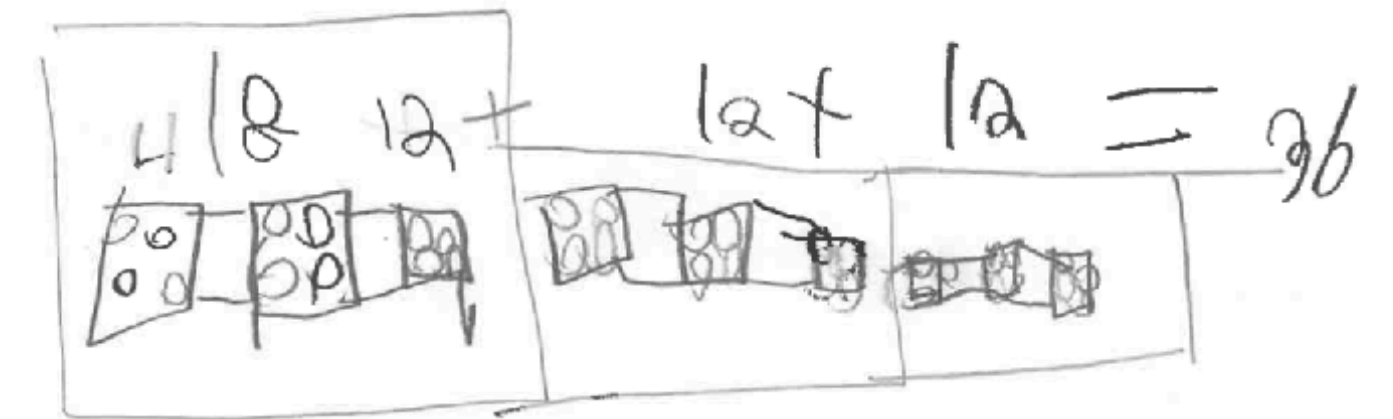
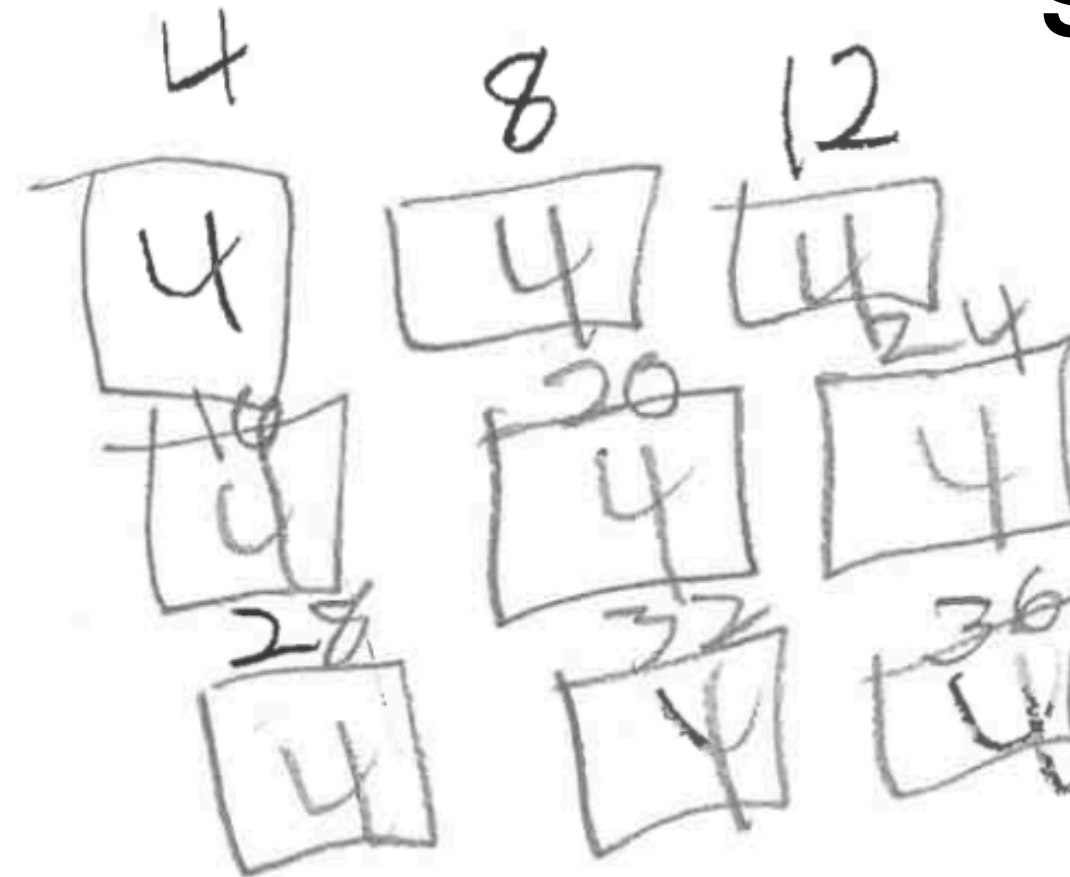
CYNTHIA



36

S7

S4



$$12 \times 3 = 36$$

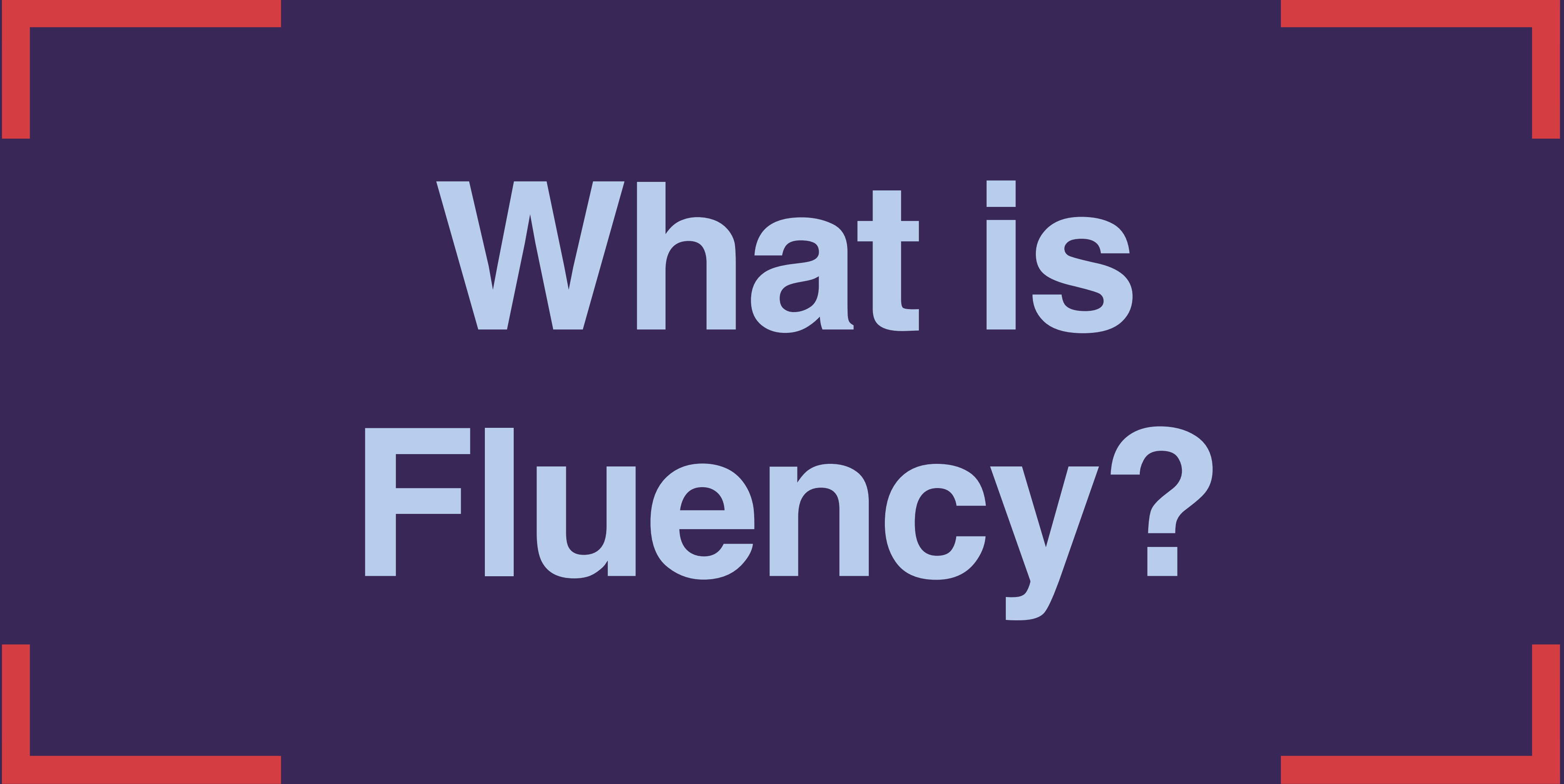
S6

Module Breakdown

Part 1: Launch the Concept

Part 2: Connect and Extend Thinking

Week 3: Building Fluency



What is Fluency?



What research says...

Adding It Up:

Procedural fluency refers to the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

(National Research Council, 2001, 121)

Principles to Action:

Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems.

(National Council of Teachers of Mathematics, 2014, 42)

Procedural Fluency in Mathematics:

Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another.

(A Position of the National Council of Teachers of Mathematics, 2014)

Accurate

Finding the right answer.

Efficient

Solving a problem with a strategy that saves time or labor.

Flexible

Knowing multiple solution paths and choosing an appropriate strategy for a particular problem.

Adding It Up:

Procedural fluency refers to the knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.

(National Research Council, 2001, 121)

Principles to Action:

Being fluent means that students are able to choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and are able to explain their approaches, and they are able to produce accurate answers efficiently. Fluency builds from initial exploration and discussion of number concepts to using informal reasoning strategies based on meanings and properties of the operations to the eventual use of general methods as tools in solving problems.

(National Council of Teachers of Mathematics, 2014, 42)

Procedural Fluency in Mathematics:

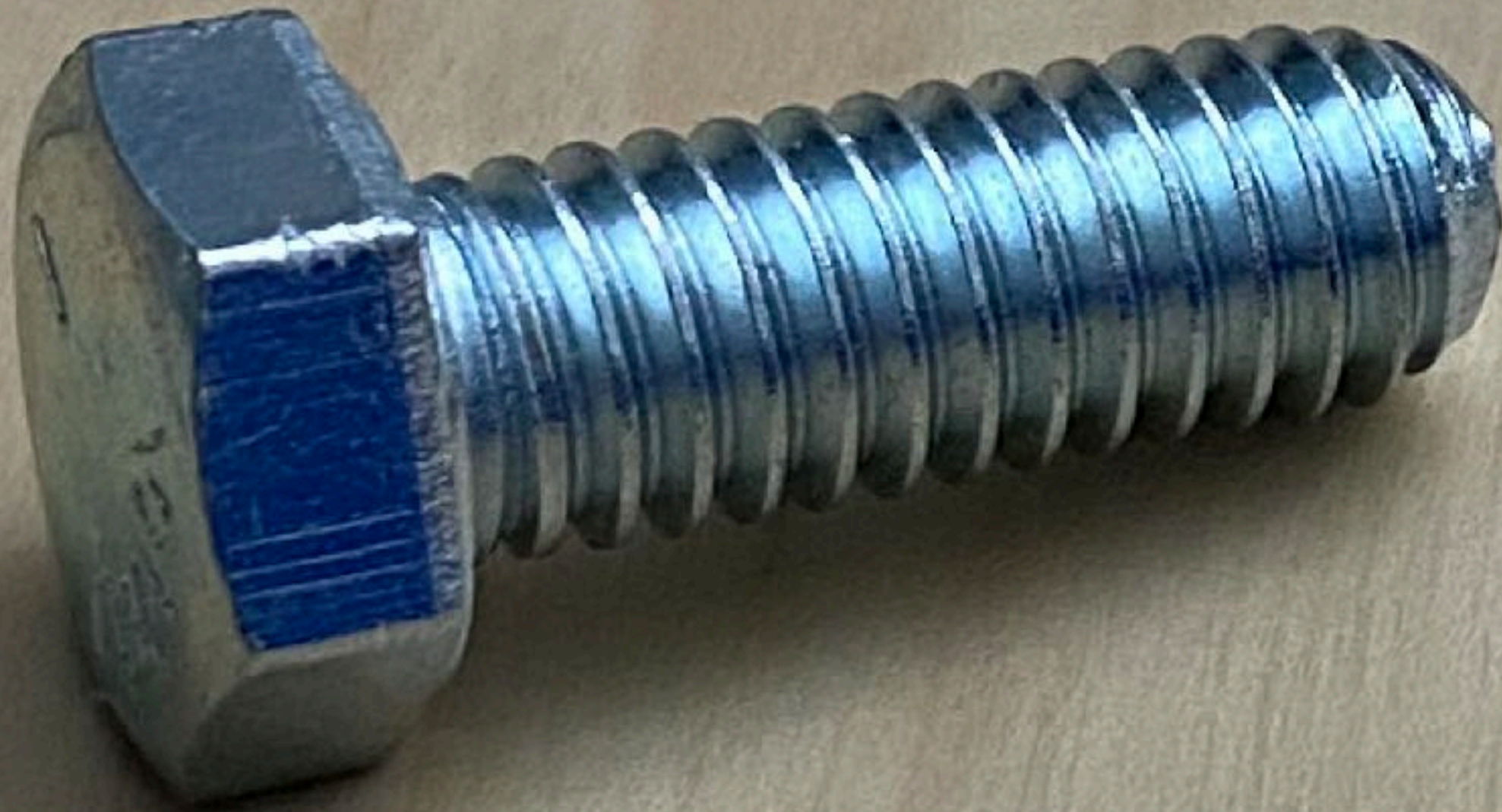
Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another.

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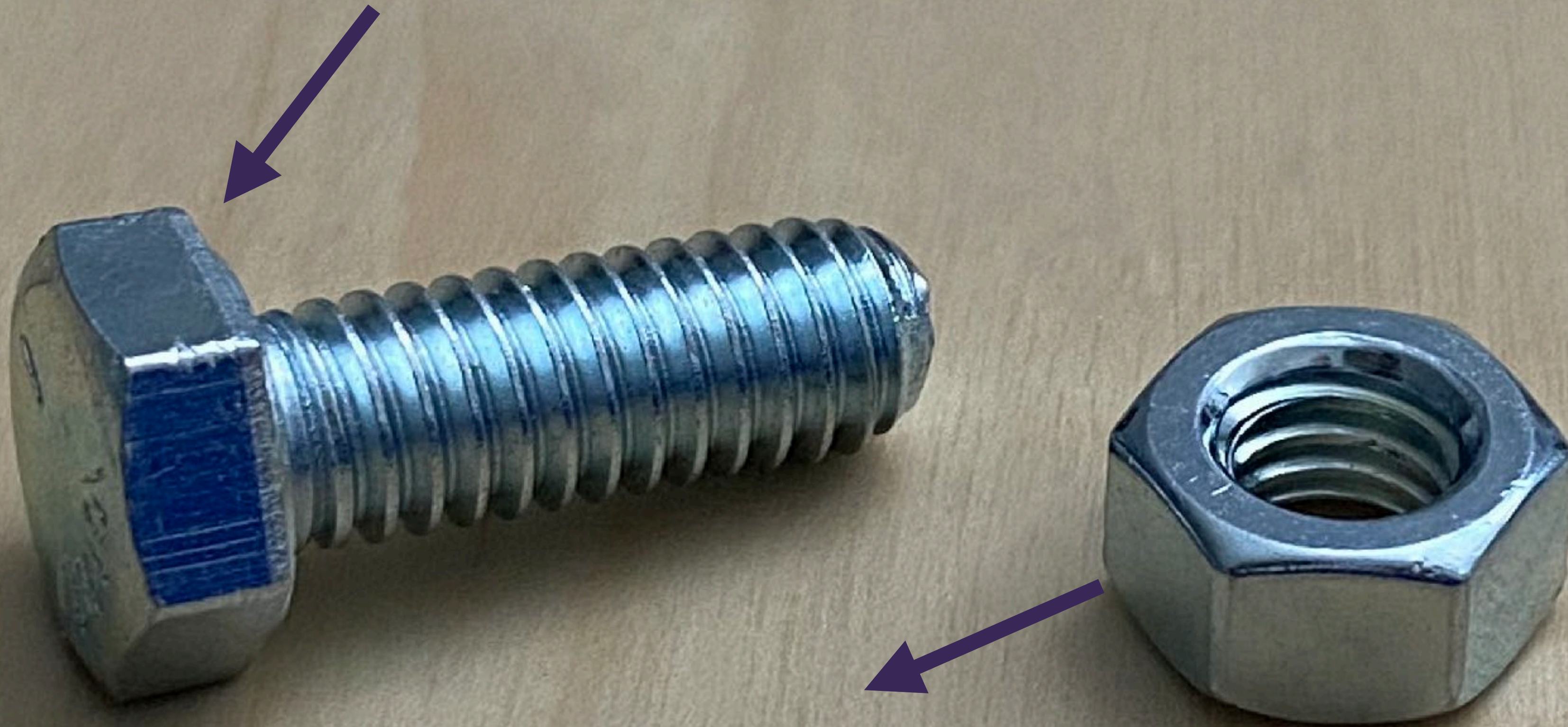
What I got right.

What I was missing.

Accuracy, Efficiency, Flexibility

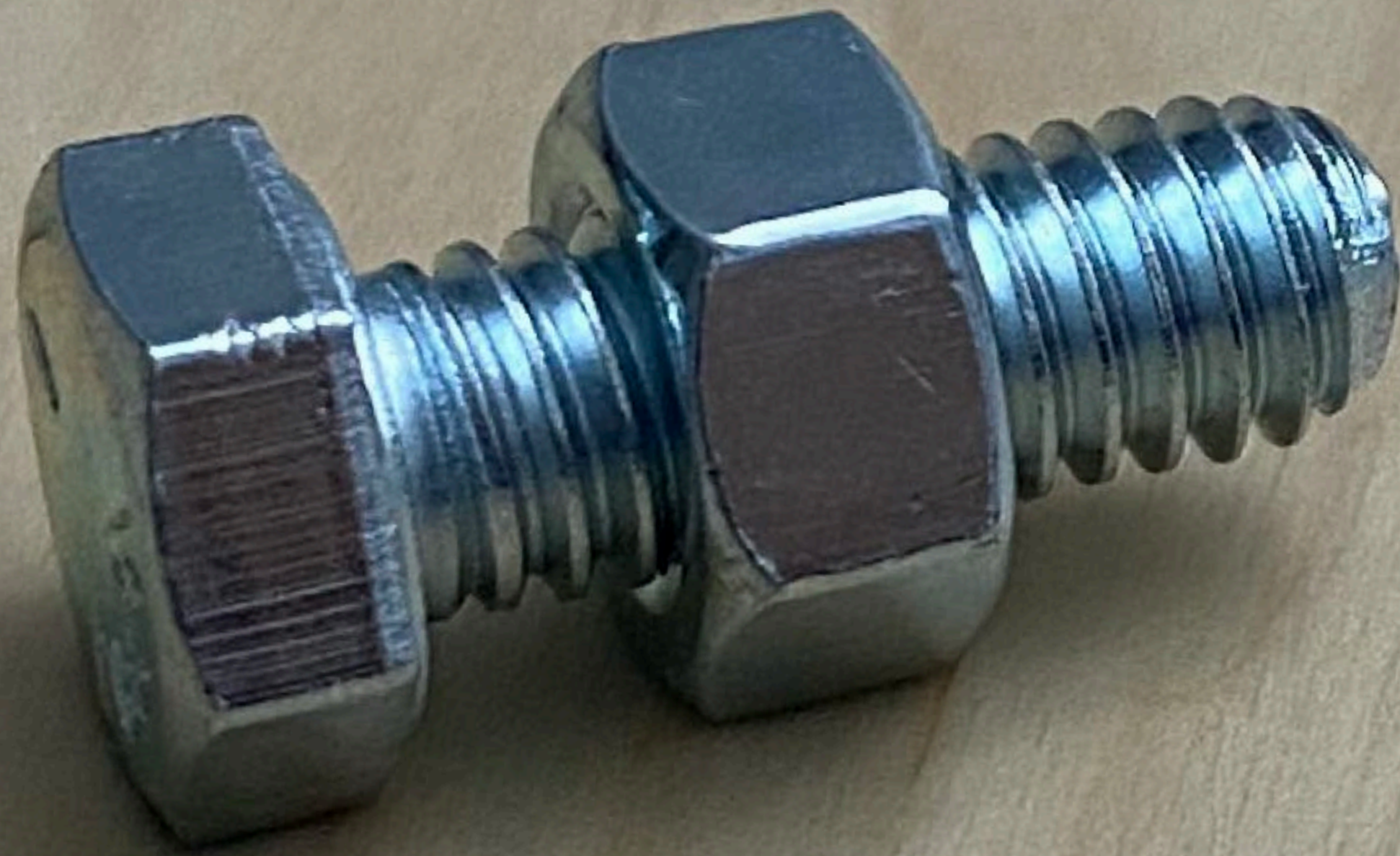


Accuracy, Efficiency, Flexibility



Context

The Nuts & Bolts of Fluency



Accuracy
Efficiency
Flexibility
Context



Lesson String





Image Talk









Images give students a visual way to think about **groups of groups, which encourage multiplicative thinking.**

Connecting Arithmetic to Algebra

The students who have the deepest trouble with middle school mathematics are those without a clear and rich set of models for what multiplication is and how it is different from addition. In the absence of such models, the students tend to rely increasingly on memorizing rules and how-tos, and their grasp is tentative and fragile.

– Bill Lancaster

Lesson String

3-Act Task
Image Talk



Contextualized Practice Problems

Name: _____ Date: _____

There are baskets with 4 peaches in each basket.
How many peaches are there in all?

[10]

[7]

[14]



Lesson String

3-Act Task

Image Talk

Contextualized Practice

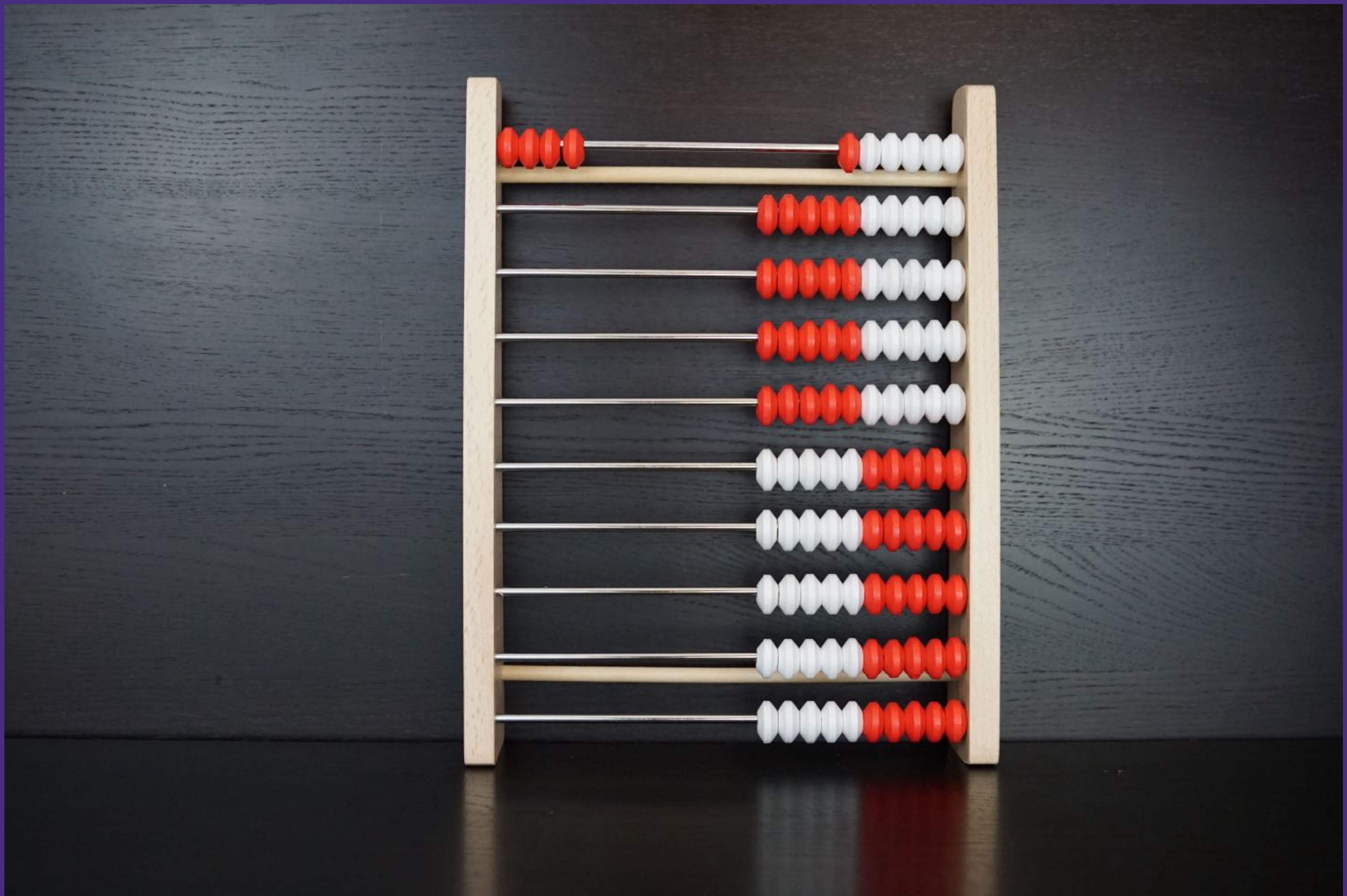


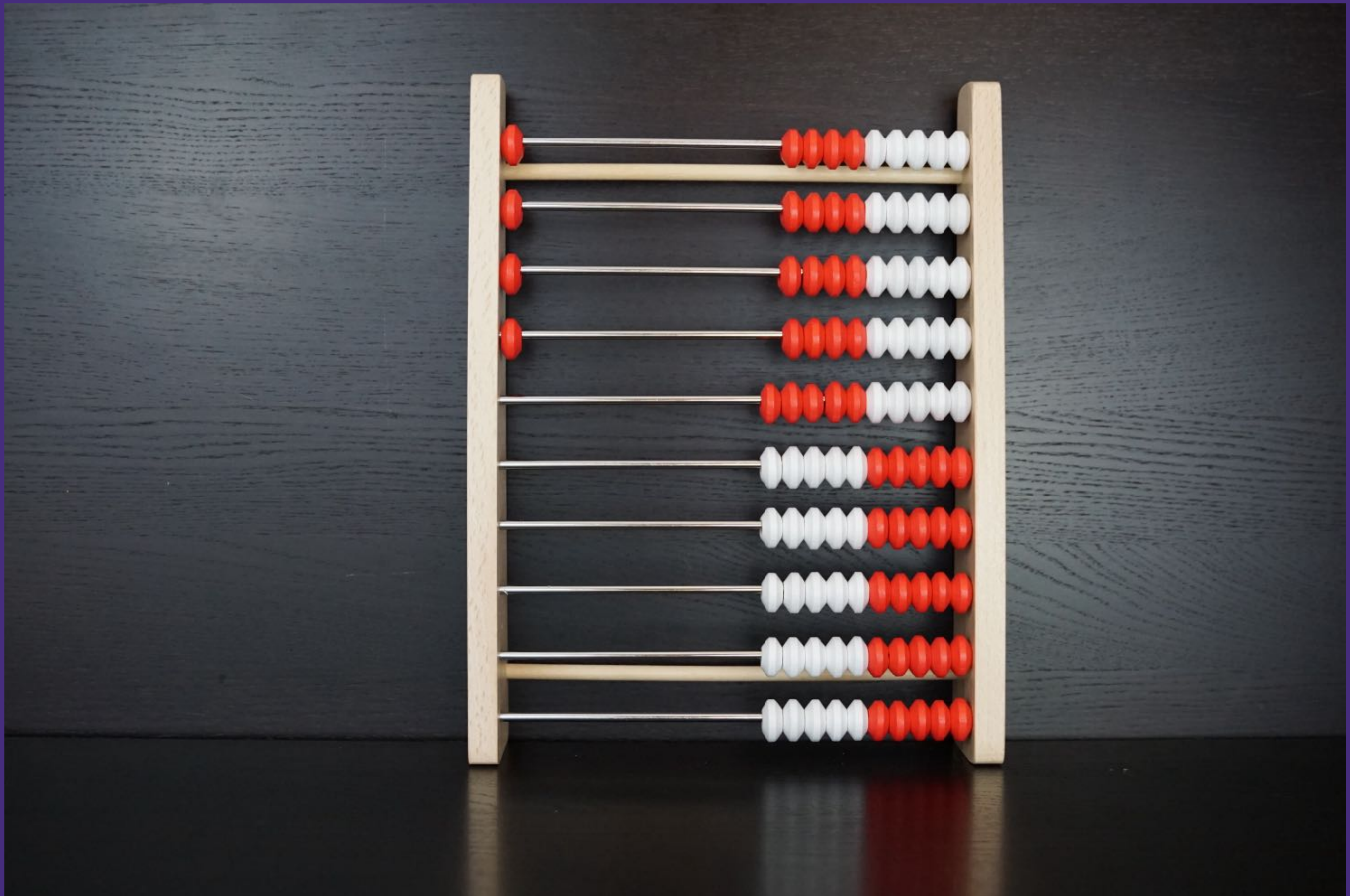
Contextualized
vs.
Decontextualized

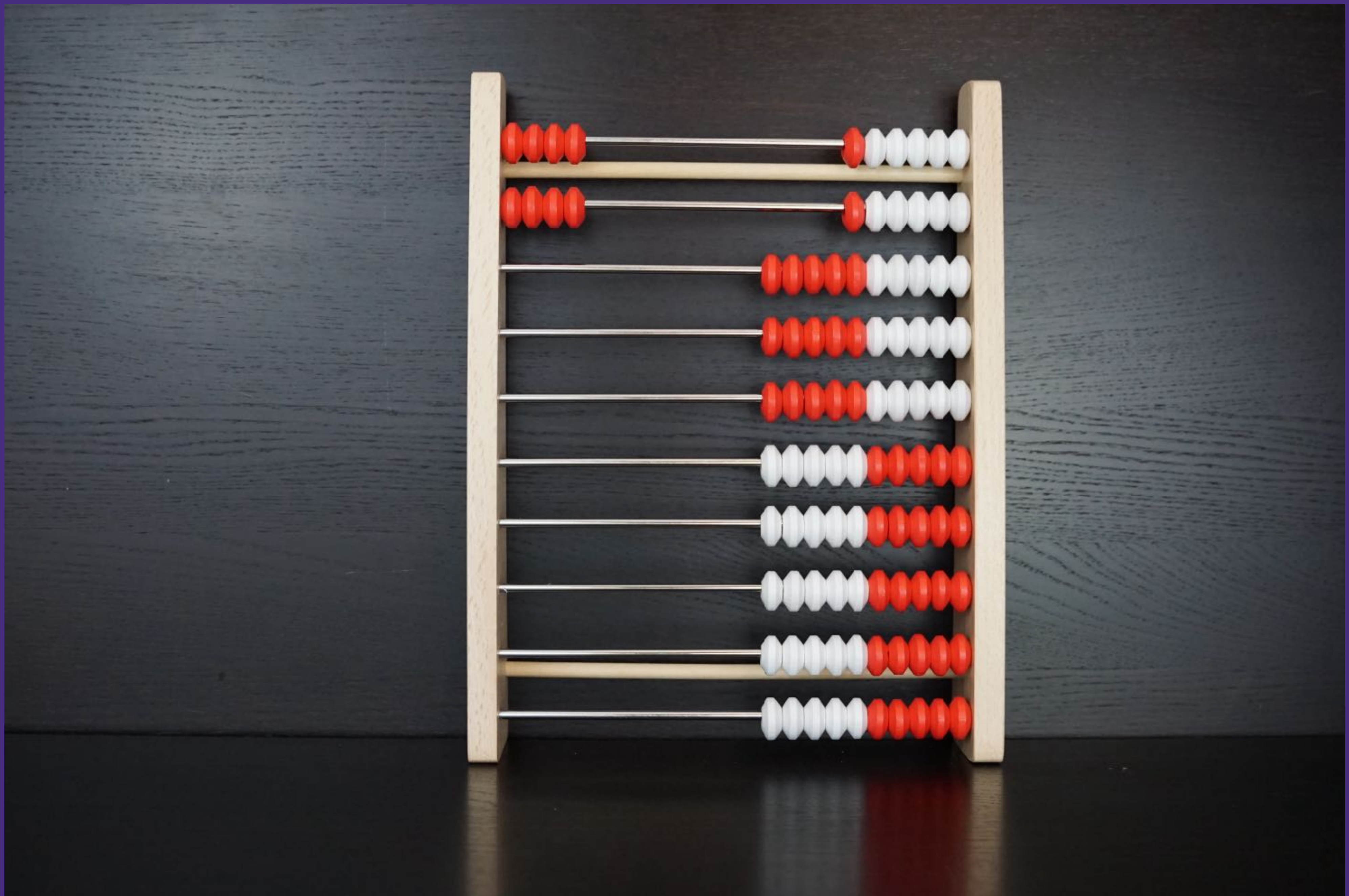


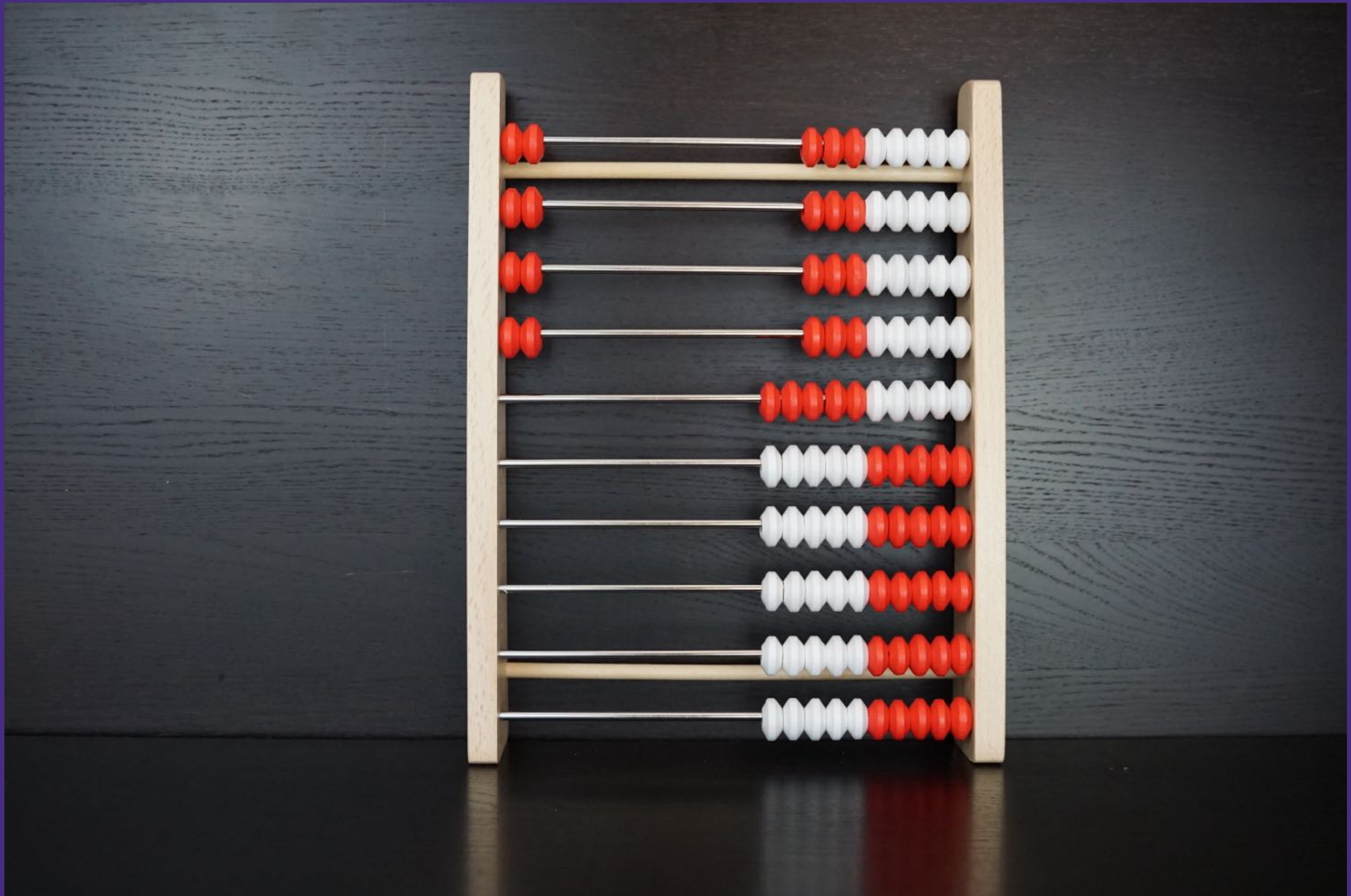
Tool Talk

#1











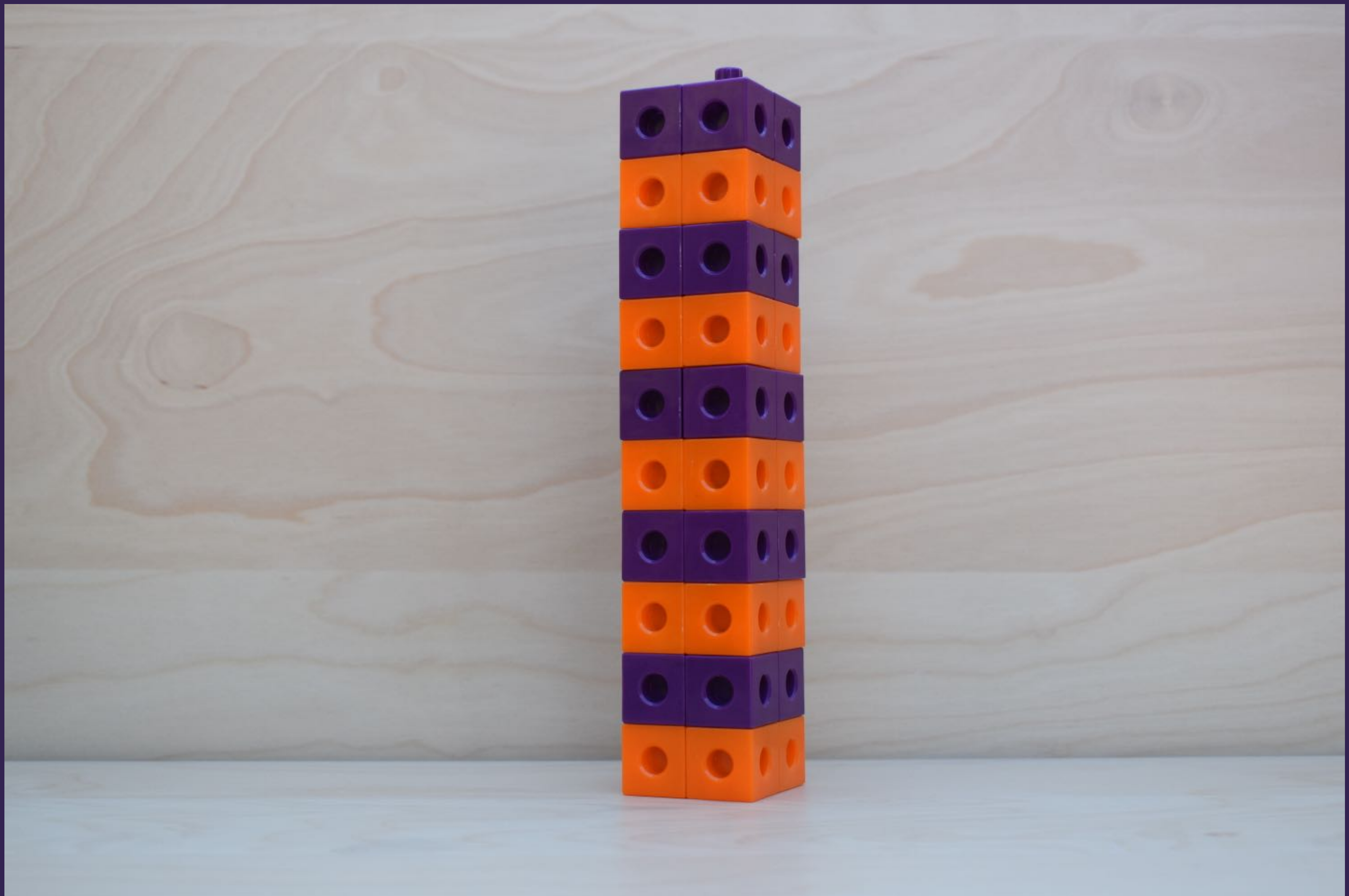
Tool Talk

#2









Lesson String

3-Act Task

Image Talk

Contextualized Practice

Tool Talk



Game

For Purposeful Practice

Lesson String

3-Act Task

Image Talk

Contextualized Practice

Tool Talk

Game



Number Talk

$$2 \times 4$$

$$4 \times 4$$

$$8 \times 4$$

$$7 \times 4$$

Number Talk

Lesson String

3-Act Task

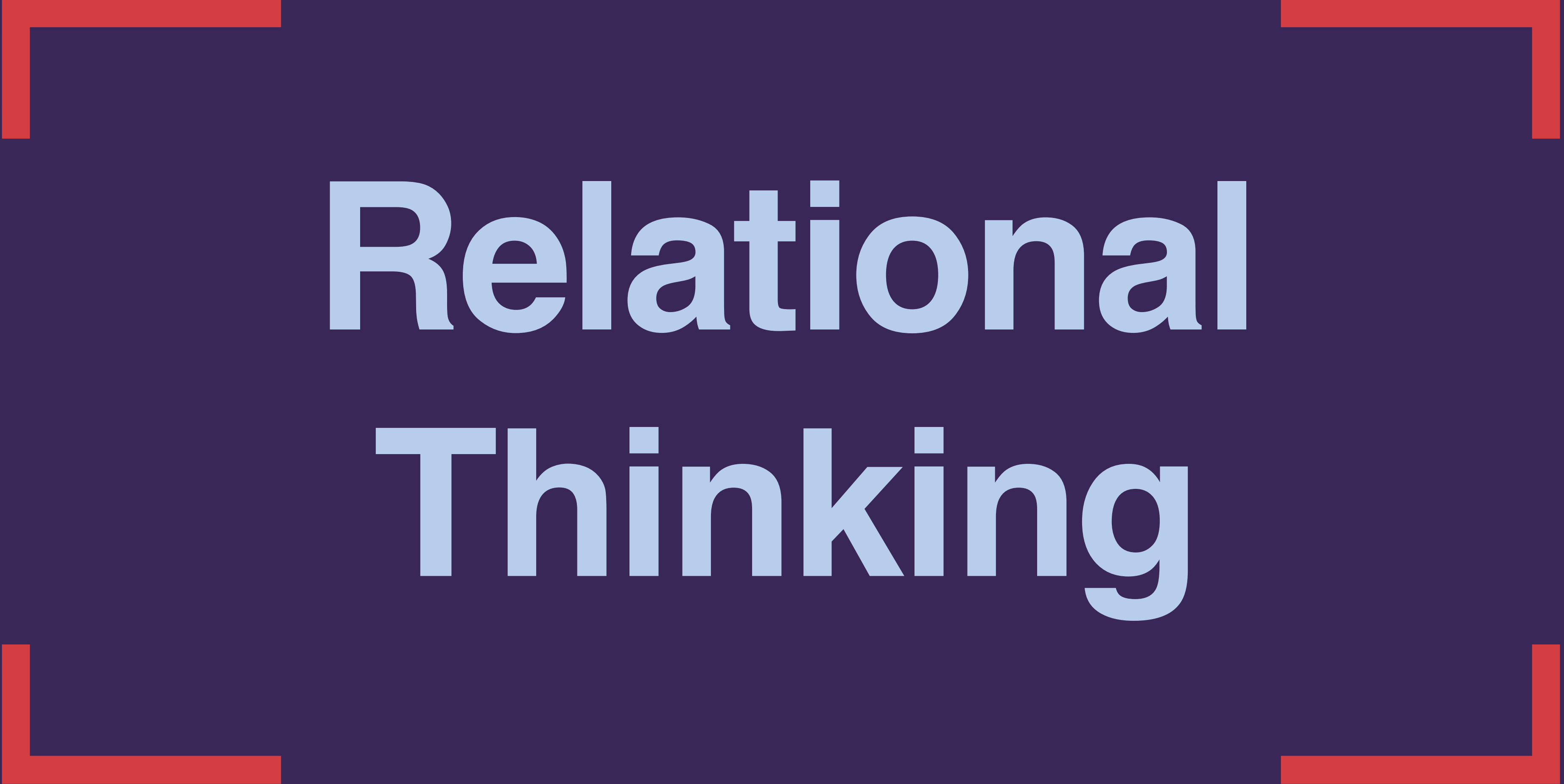
Image Talk

Contextualized Practice

Tool Talk

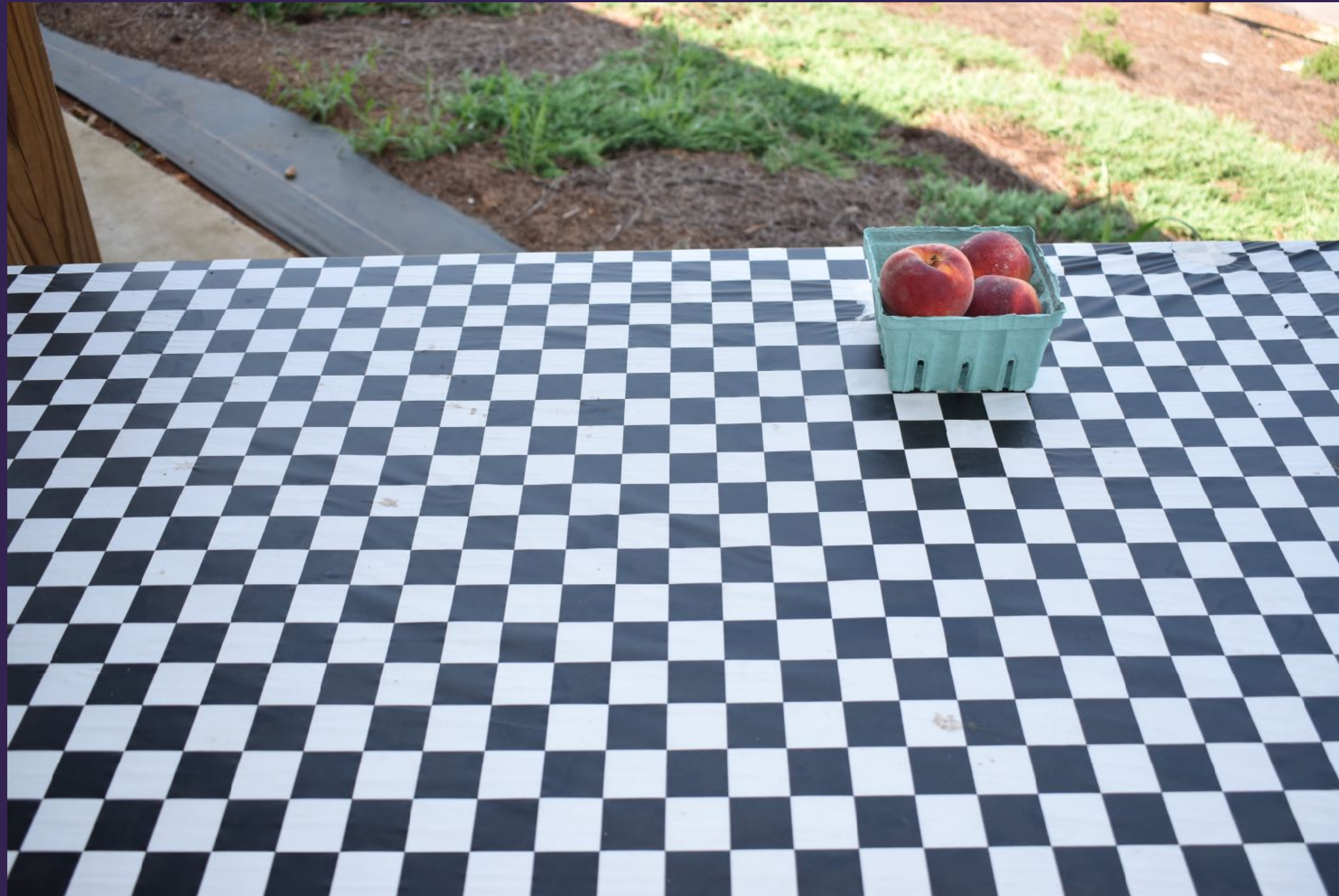
Game

Number Talk



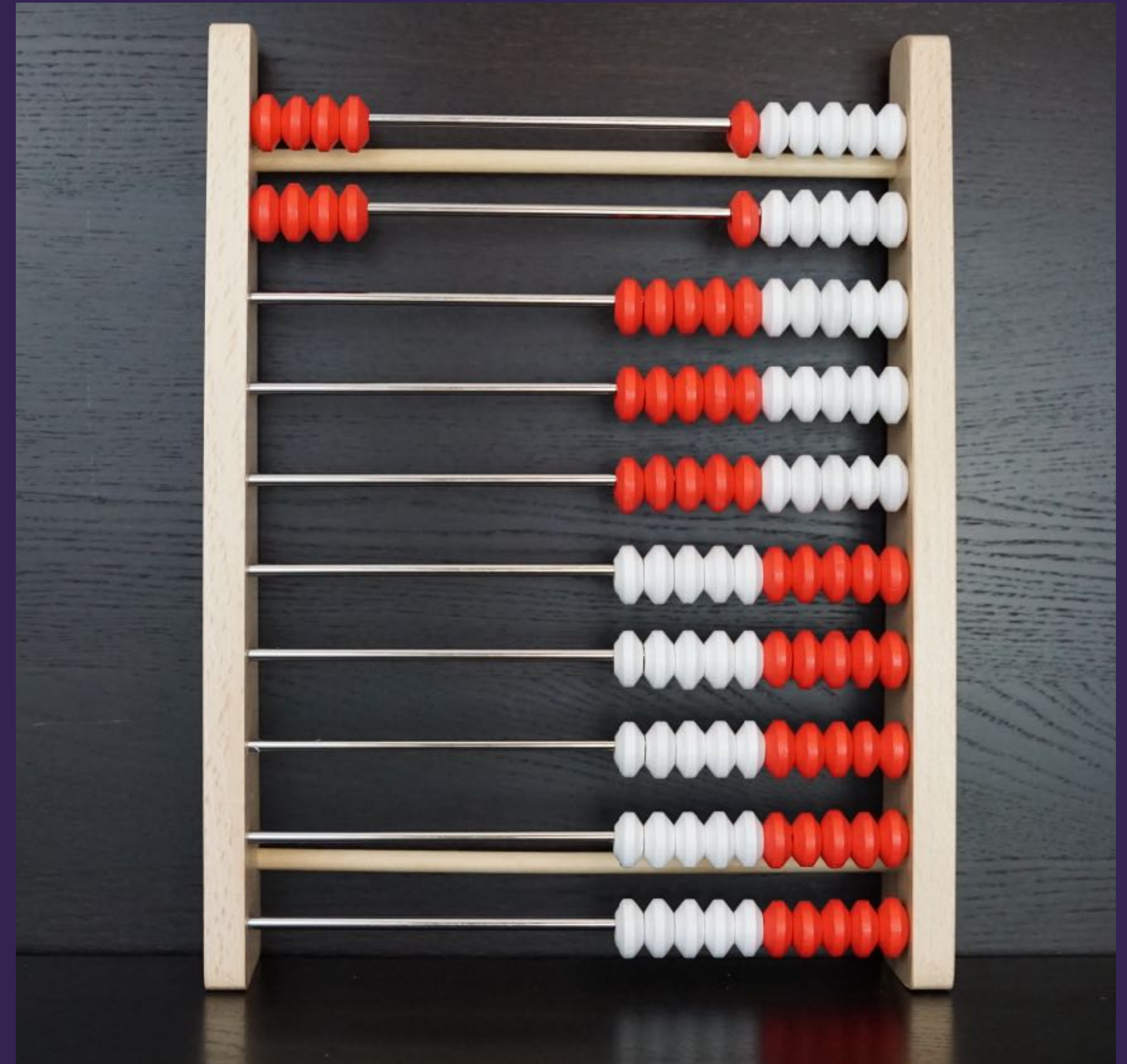
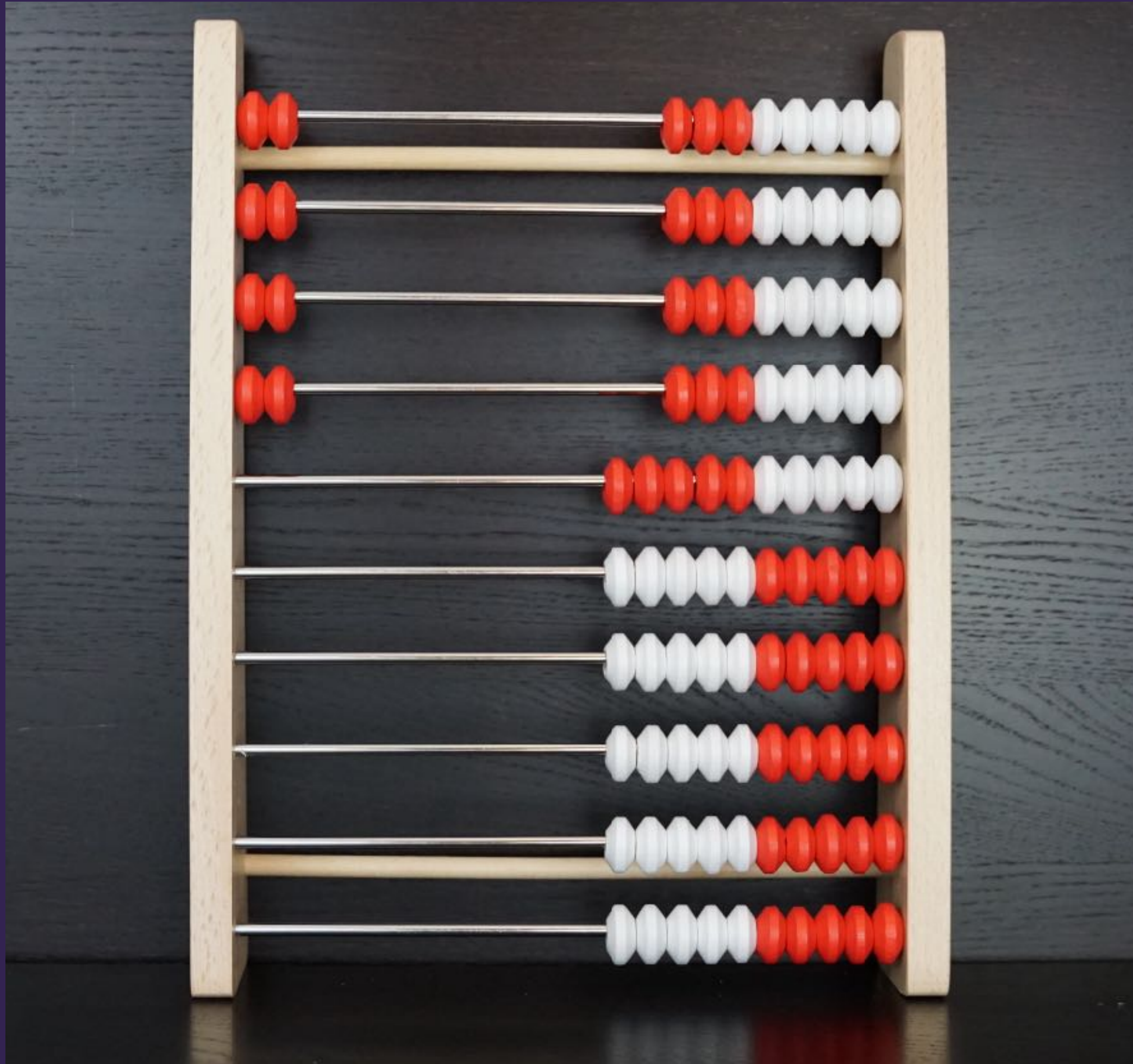
Relational Thinking

Identity Property



$$1 \times 4 = 4$$

Commutative Property



$$4 \times 2 = 2 \times 4$$

Associative Property



$$(2 \times 2) \times 3 = 2 \times (2 \times 3)$$

Distributive Property

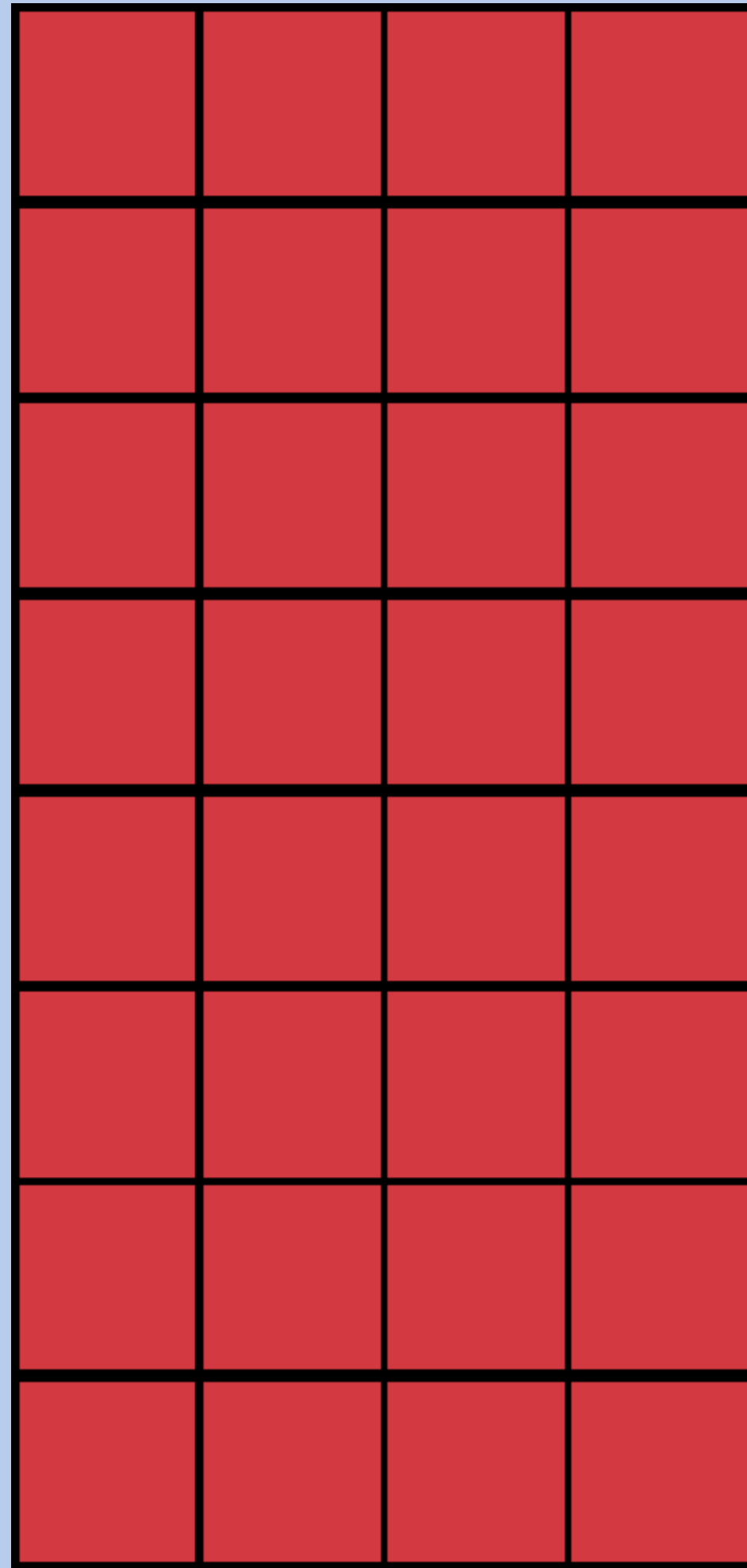


$$8 \times 4 = (6 \times 4) + (2 \times 4)$$

Reasoning that is Scalable



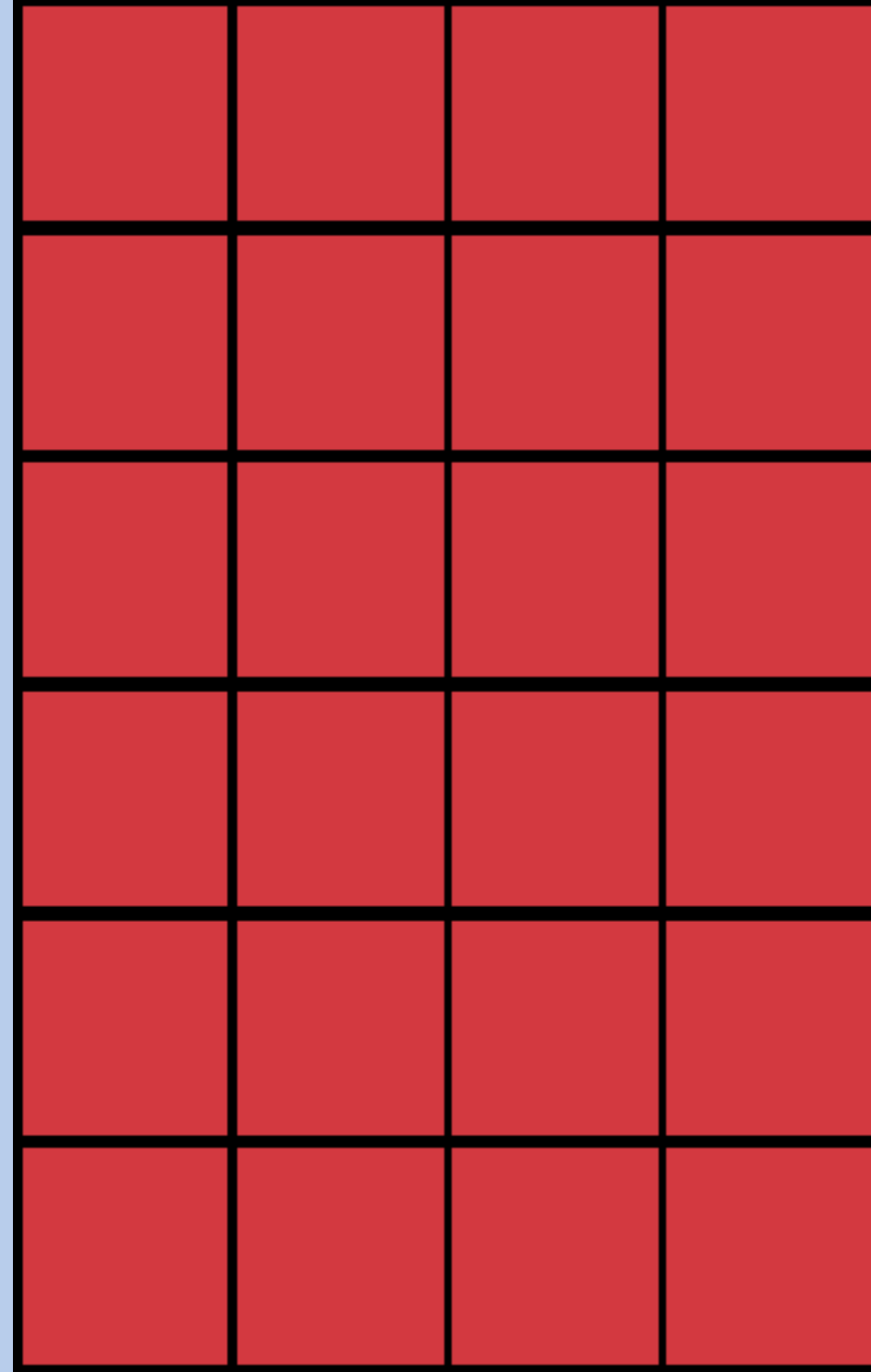
4



8

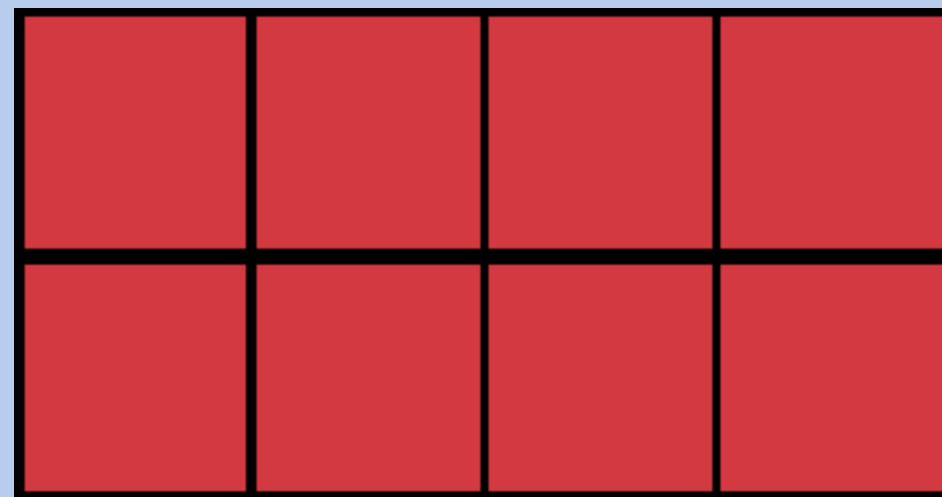
$$8 \times 4$$

4

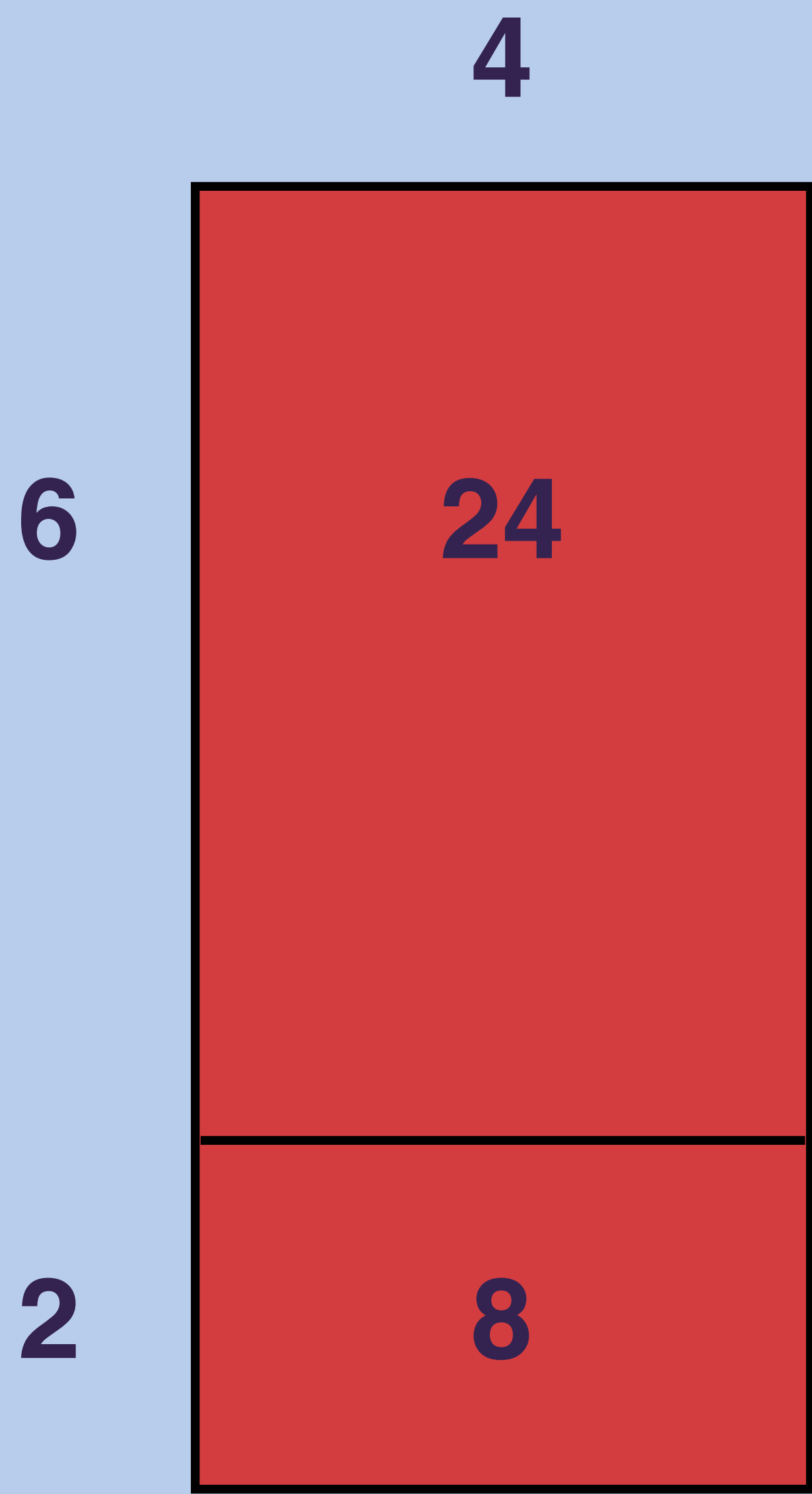


6x4

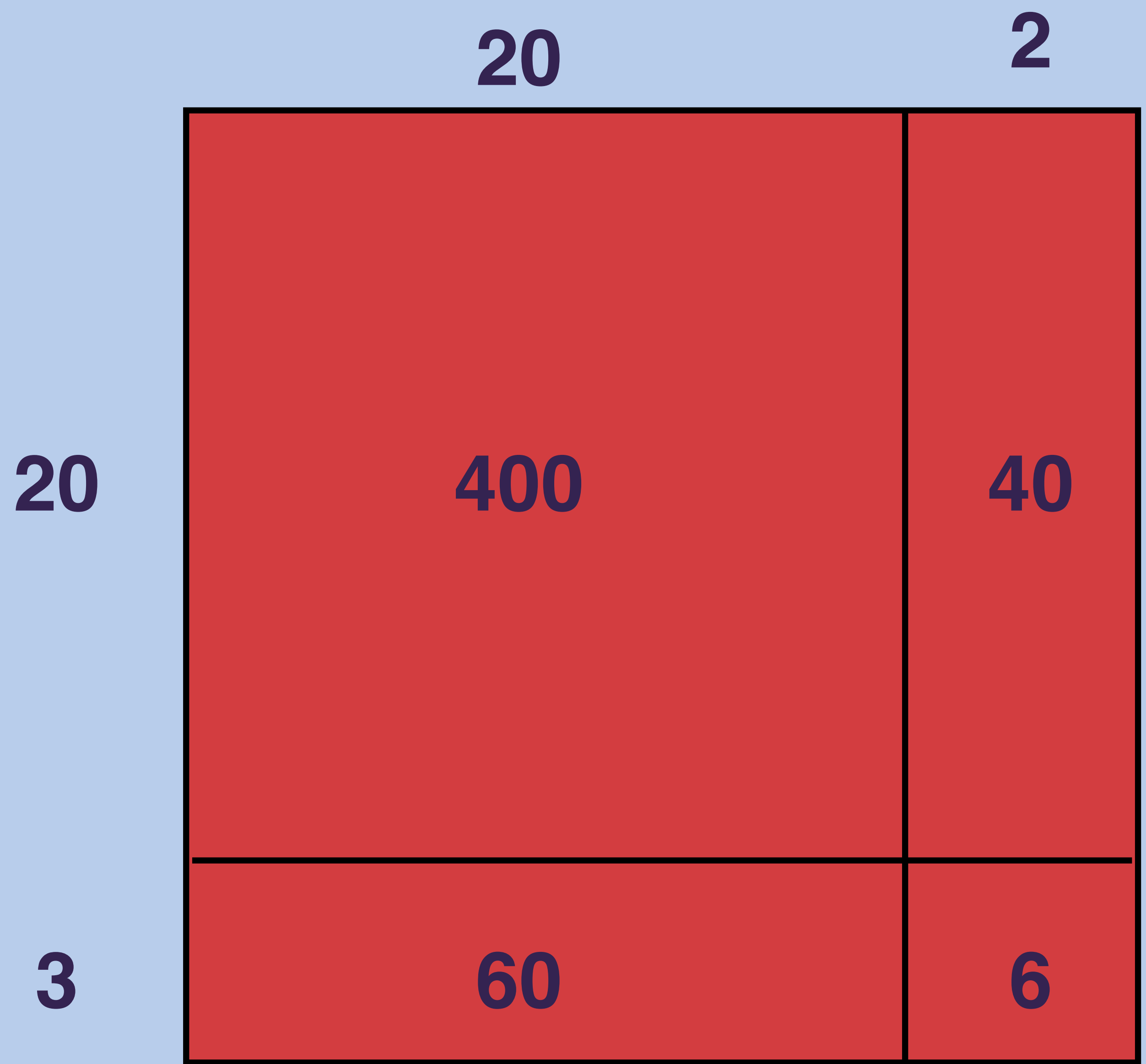
8 x 4



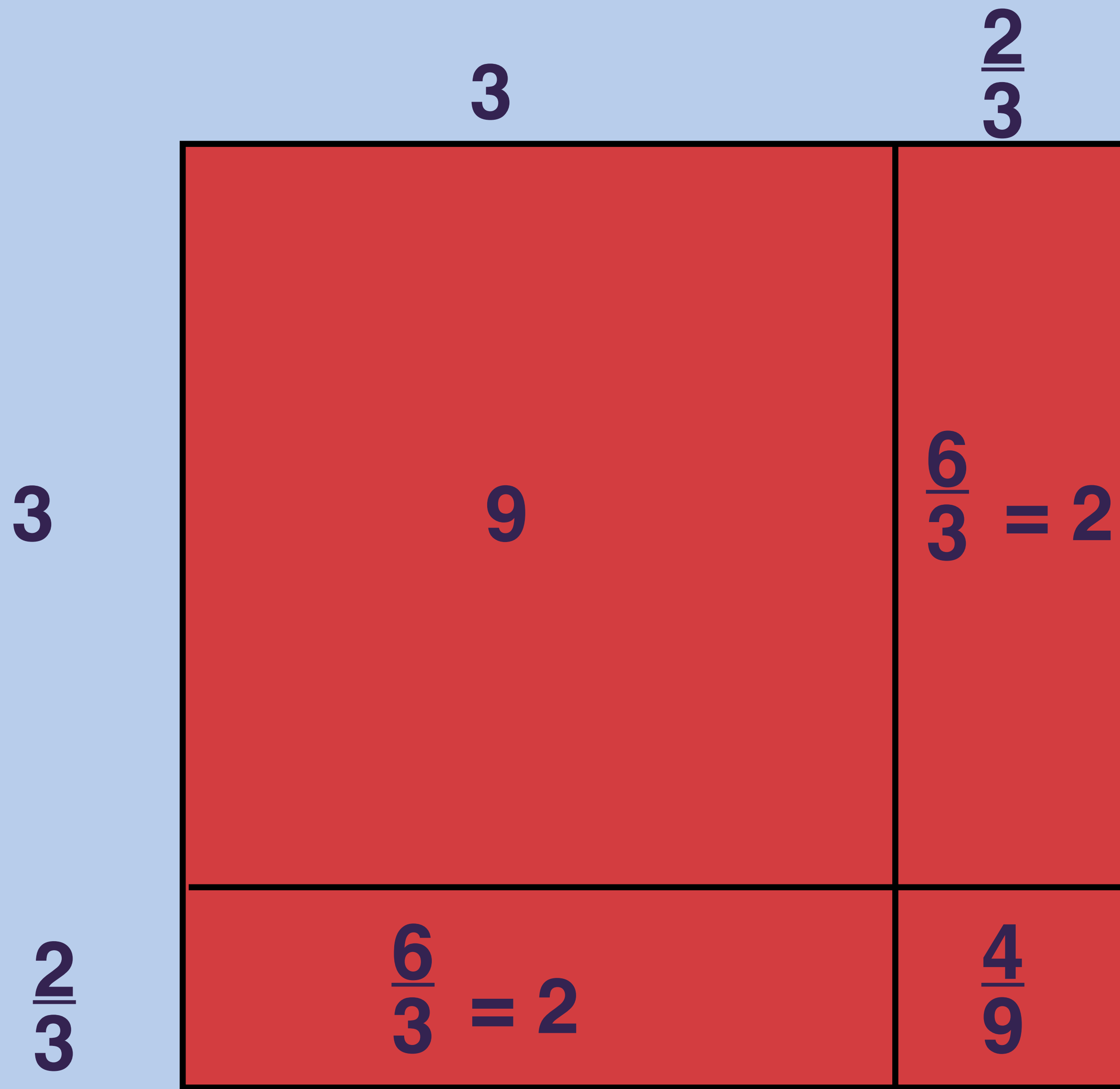
2x4



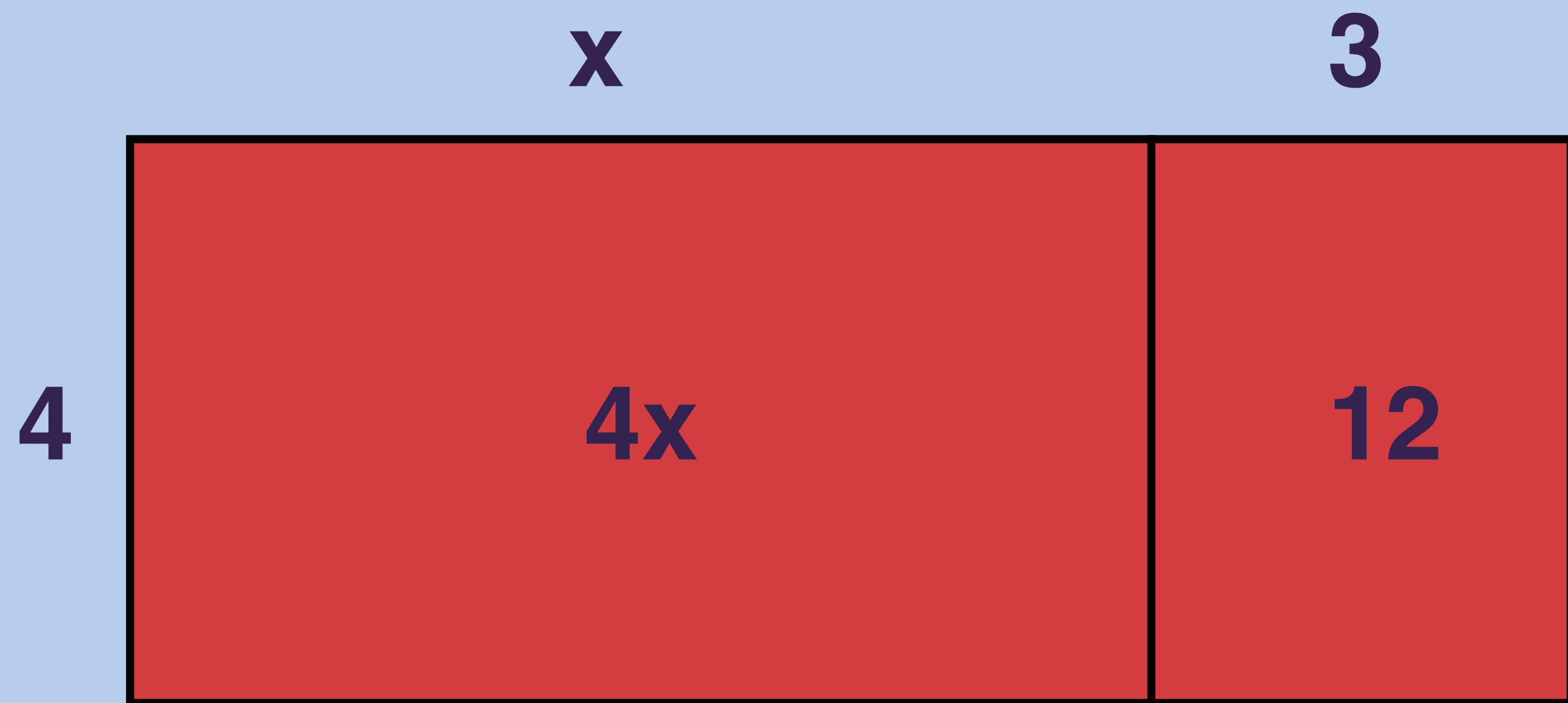
$$8 \times 4$$



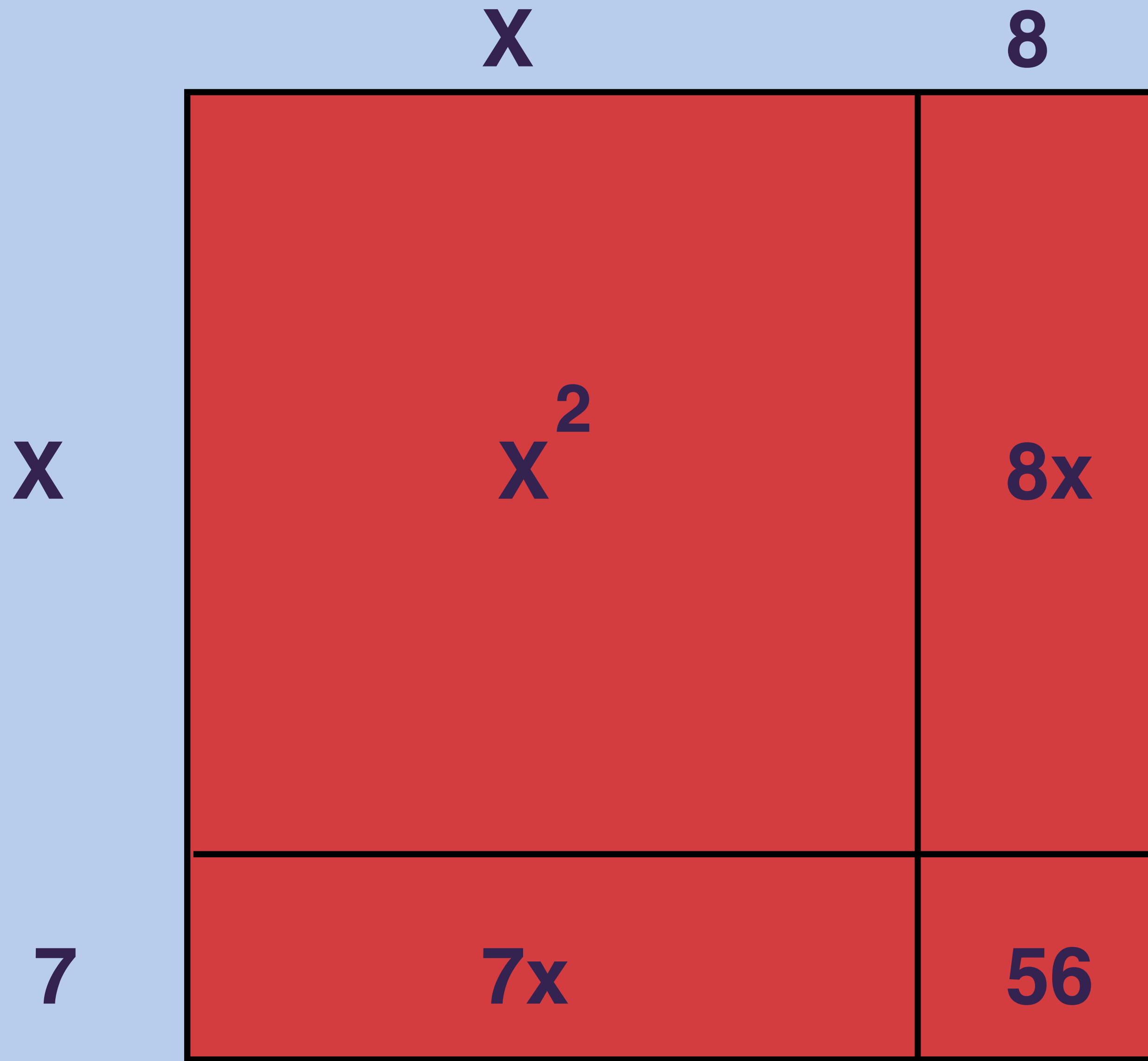
23 x 22



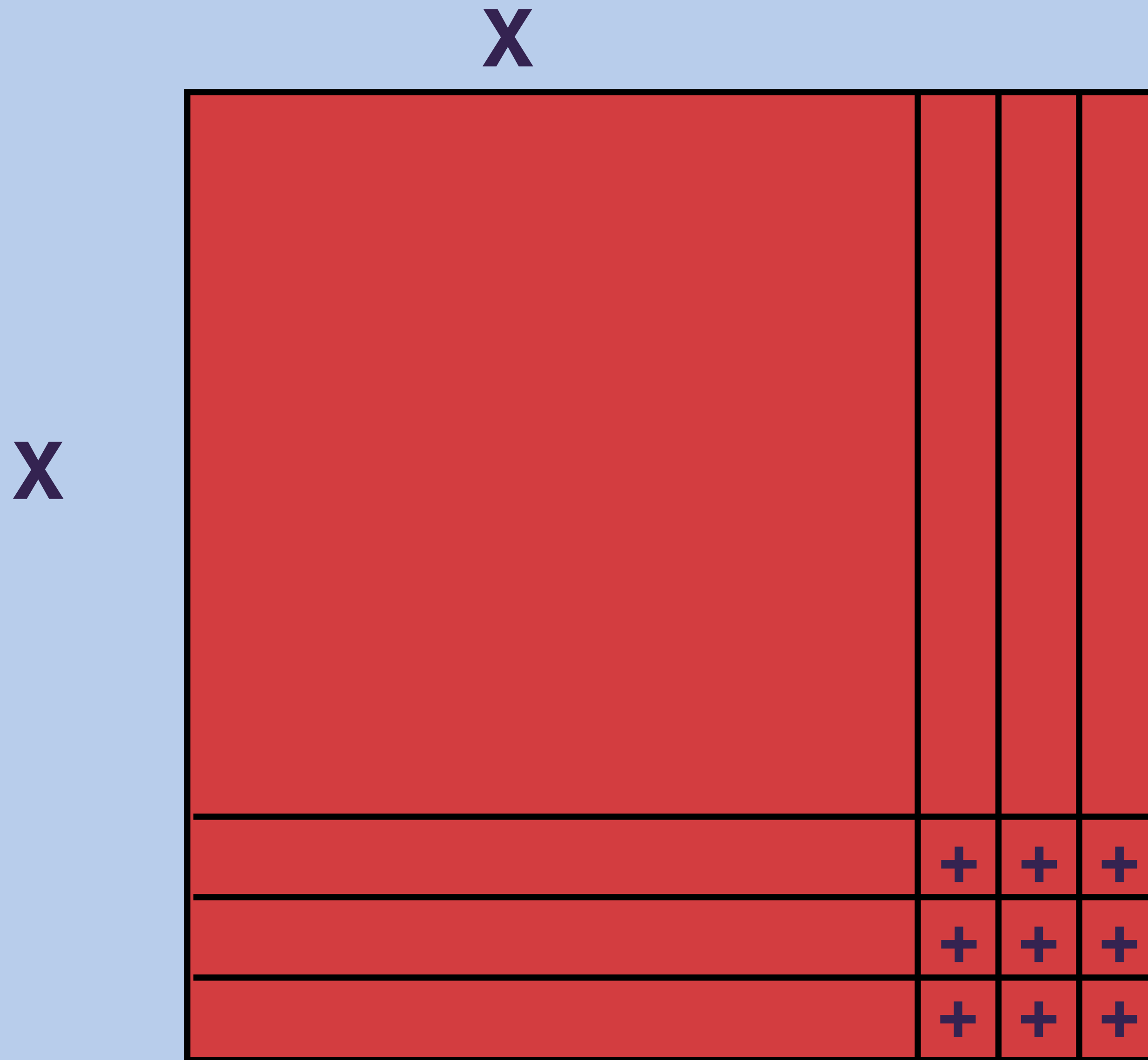
$$3\frac{2}{3} \times 3\frac{2}{3}$$



$$4(x + 3)$$

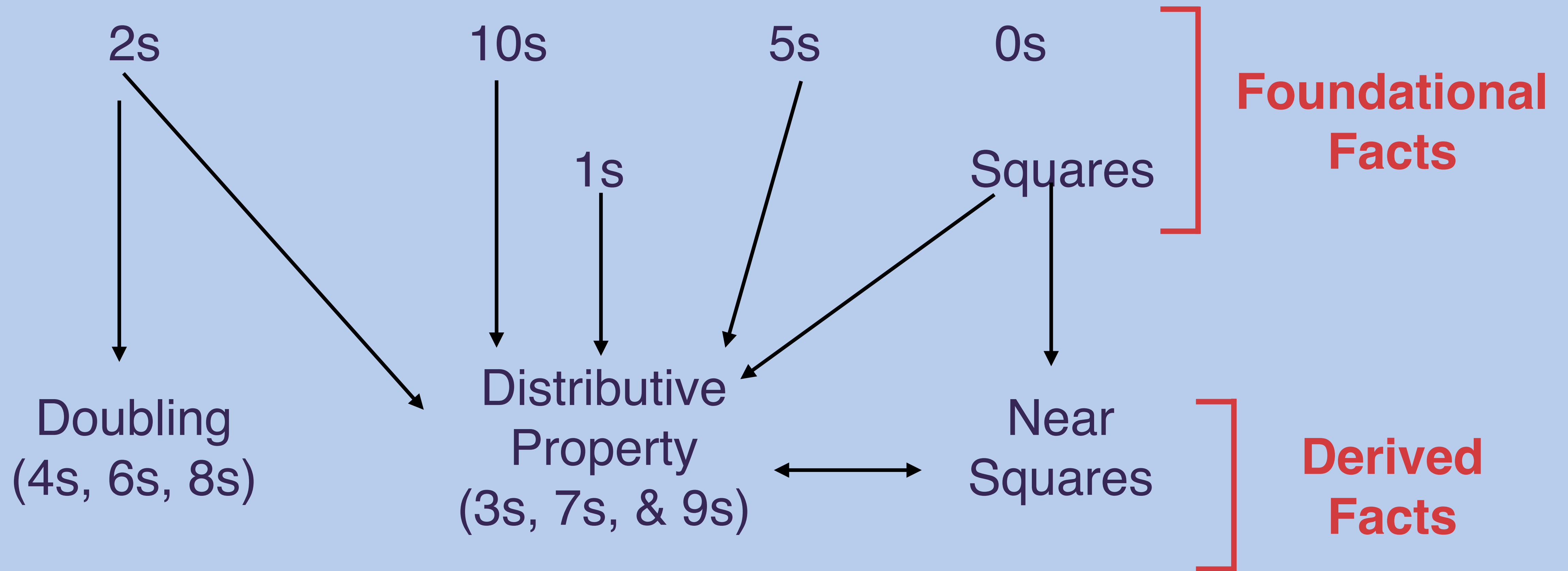


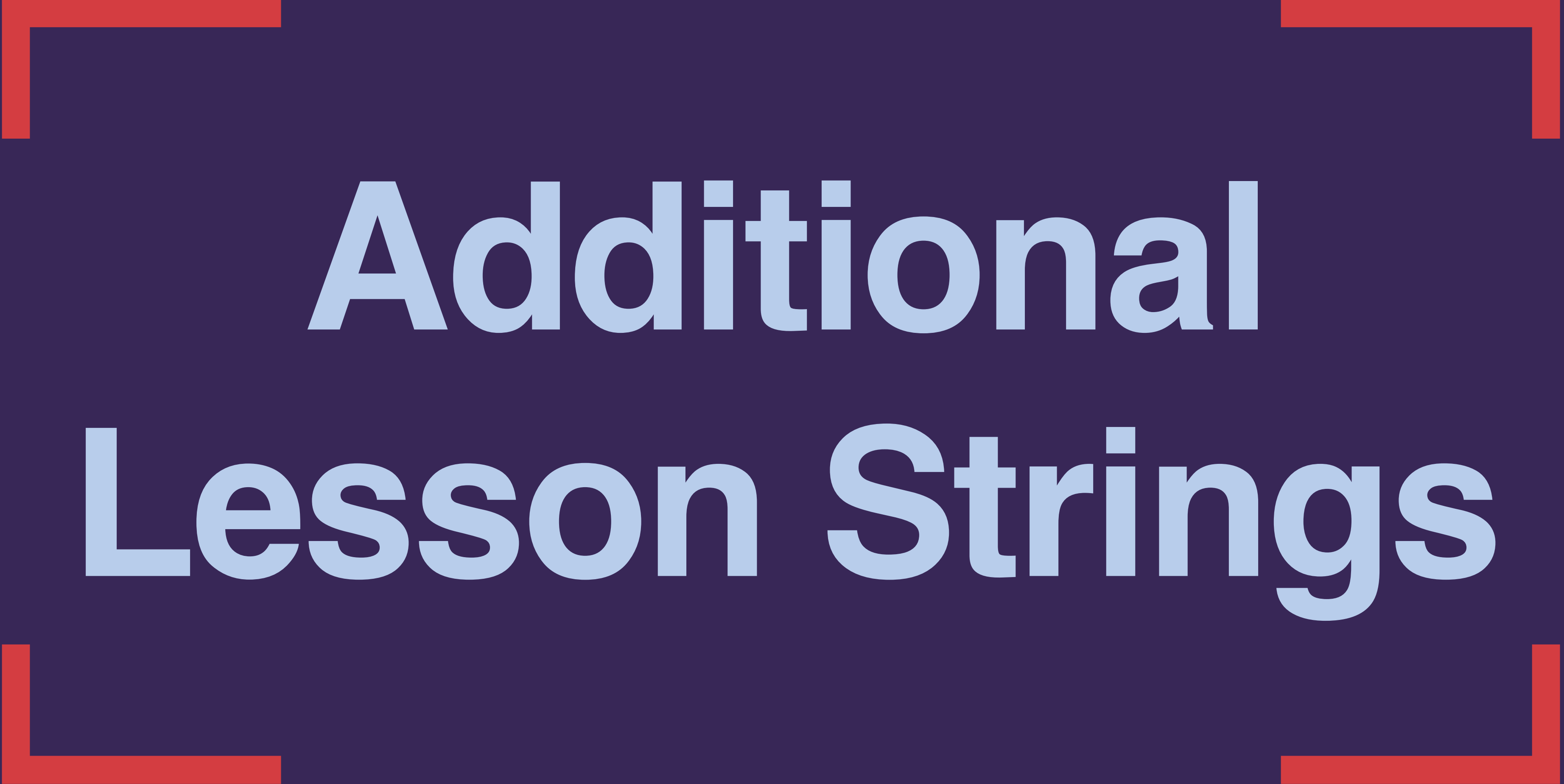
$$(x + 8)(x + 7)$$



$$(x+3)^2 = x^2 + 6x + 9$$

Multiplication and Division Fact Fluency





Additional Lesson Strings



Pause
||

Go Play

If you want

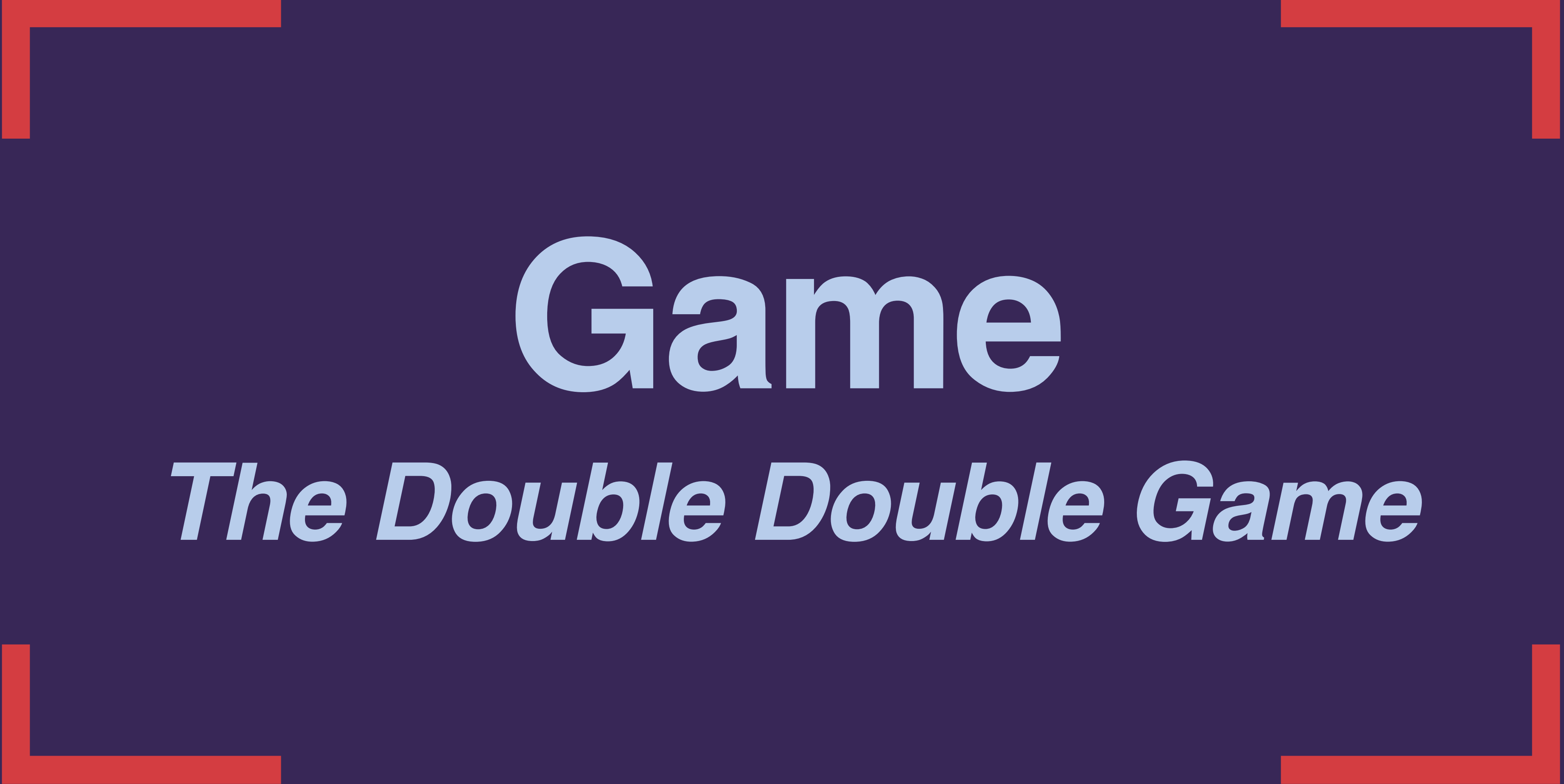


Module Breakdown

Part 1: Launch the Concept

Part 2: Connect and Extend Thinking

Week 3: Building Fluency



Game

The Double Double Game

The Double Doubles Game

(6-Sided Die)

6

6	8	16	20	3	12
12	10	4	8	24	8
1	6	24	2	5	20
8	3	12	6	12	16
5	16	8	4	2	8
4	10	1	8	12	3

Place the chip in the box for the move you want to make.

Identity Property	Double	Double the Double
-------------------	--------	-------------------





Game

Collecting Quotients

Collecting Quotients (Dividing by 4)



Roll	Choose a Dividend		
	20	12	36
	8	16	32
	40	4	24
	32	20	28
	4	36	8
	28	40	12
	8	24	16
	24	20	32
	36	4	28
	16	12	40

PLAYER A _____		PLAYER B _____	
Equation	Quotient	Equation	Quotient
$24 \div 4 = 6$	6	$36 \div 4 = 9$	9
Total Quotients Collected		Total Quotients Collected	



Capturing Rectangles

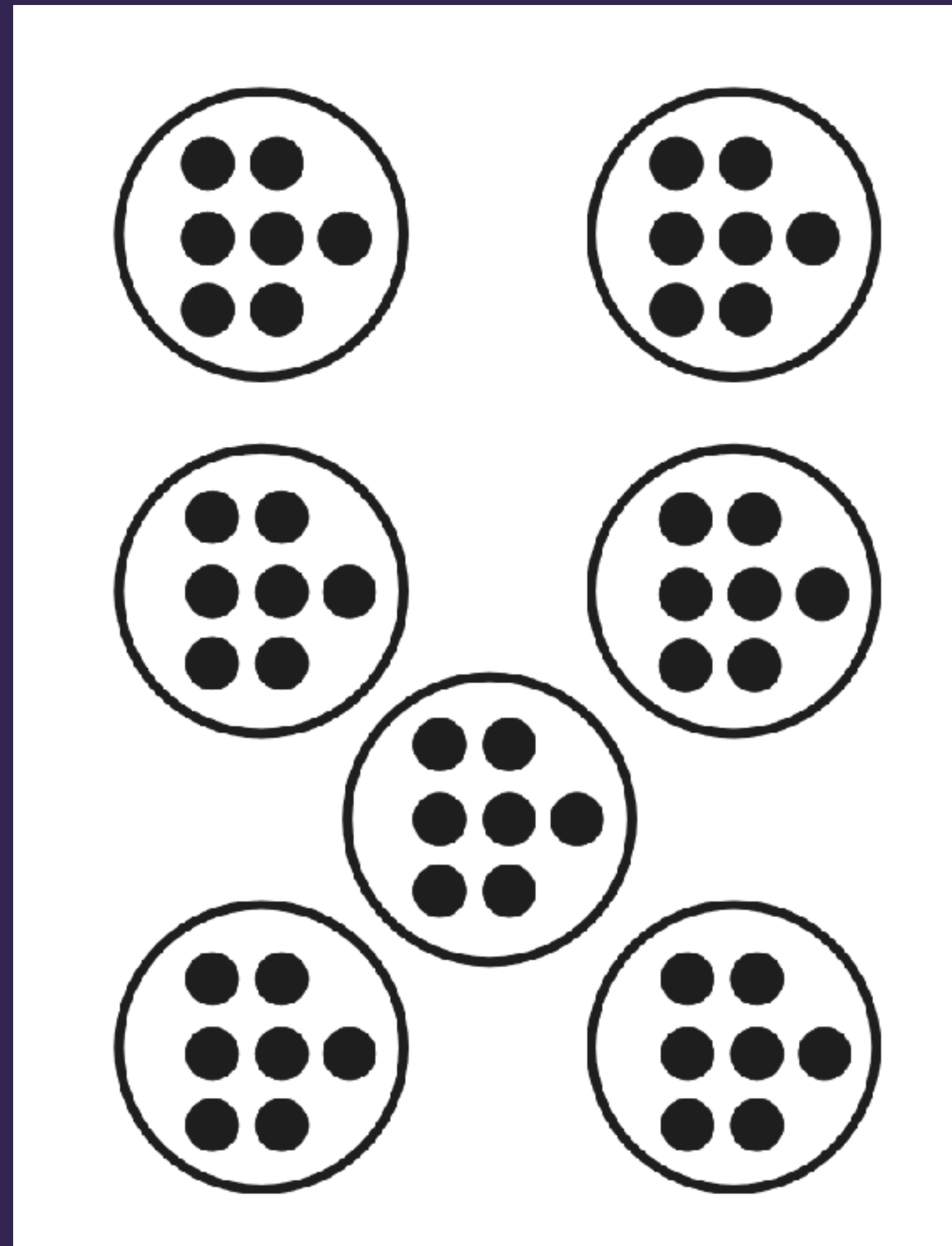


Building Equations

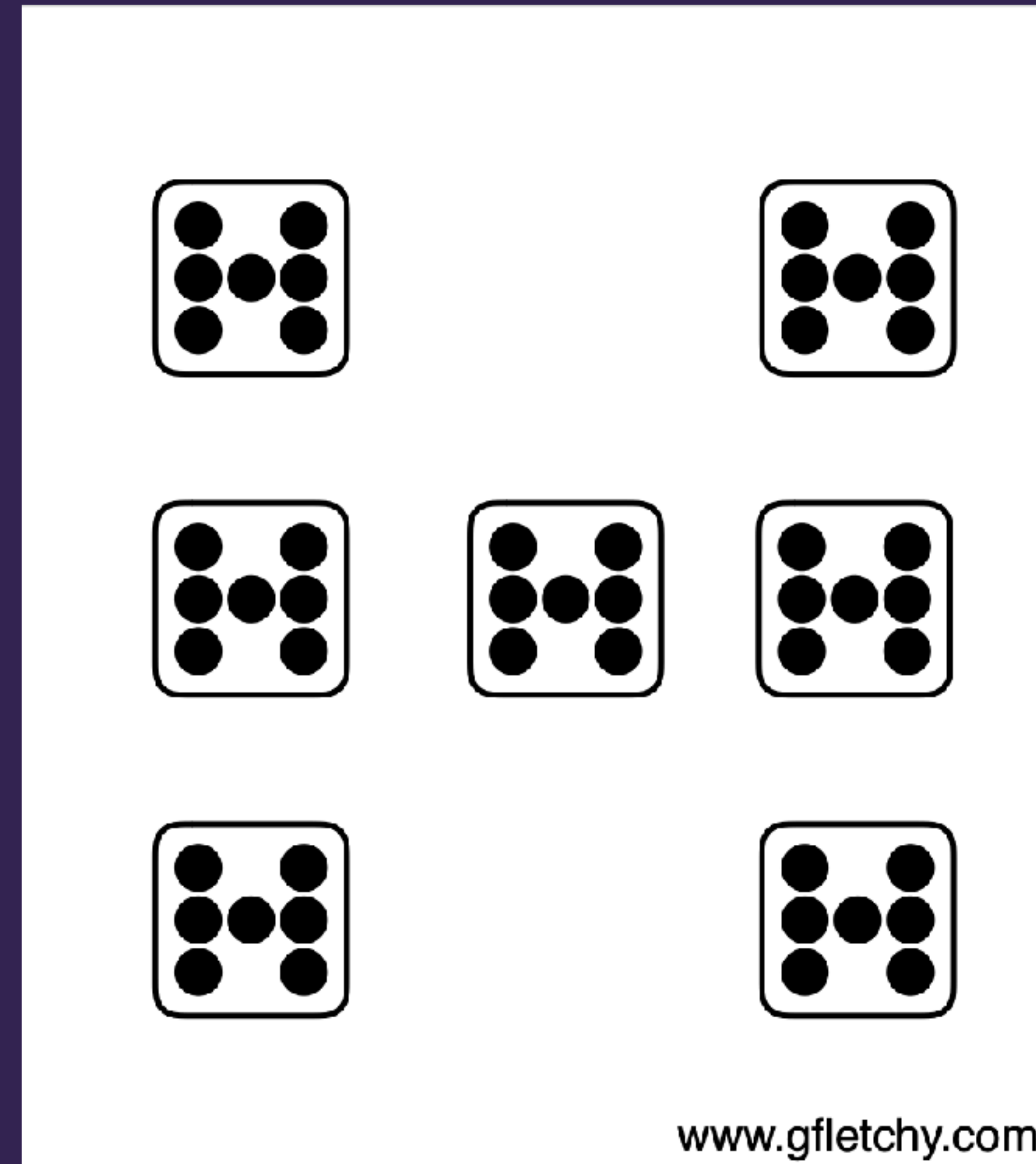


Multiplication Subitizing Cards

Old Version



New Version





Berkely Everette

Math Flips



Open Middle Problem

Open Middle

Directions: Using the digits 1-9, fill in the boxes below to complete the equations. Each box must be used and a digit can only be used one time.

$$(\square \times 4) + (\square \times 4) = \square \times \square$$

$$(\square \times 4) + (\square \times 4) = 4 \times \square$$

Pause
II

Open Middle

Directions: Using the digits 1-9, fill in the boxes below to complete the equations. Each box must be used and a digit can only be used one time.

$$\left(\boxed{5} \times 4 \right) + \left(\boxed{1} \times 4 \right) = \boxed{4} \times \boxed{6}$$

$$\left(\boxed{7} \times 4 \right) + \left(\boxed{2} \times 4 \right) = 4 \times \boxed{9}$$



Coming Up...

Week 1: Whole Number and Place Value

Week 2: Building Fluency through 1000 (Add & Sub)

Week 3: Building Fluency through 100 (Mult & Div)

Week 4: Building Fluency through 1000 (Mult & Div)

Week 5: Extending Place Value and Operations (Decimals)

Week 6: Number Statistical Reasoning



Foundations of Numerical Reasoning in Grades 3-5

Module 3: Building Fluency Through 100 (Mult & Div)

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