Using Progressions and Learning Trajectories to Guide Intervention in Multiplication and Division

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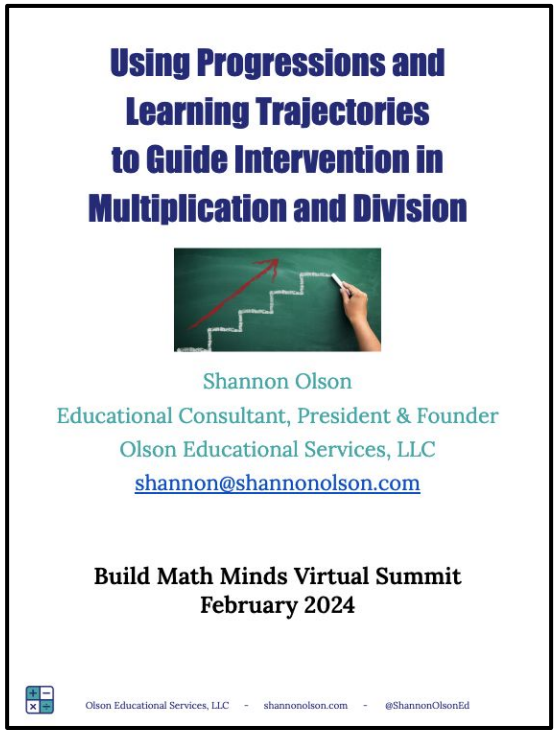
Build Math Minds Virtual Summit, February 2024

Welcome

Setting Up for the Session



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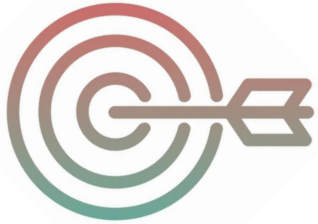
Learning Intentions:



- **anticipate** student thinking for **multiplication and division**
- observe and describe **student thinking**
- understand **learning progressions** and **learning trajectories**
- **make connections** for **planning intervention**

Learning Intentions:

- **anticipate** student thinking fo**ltiplication and division**
- observe a**ing**
- unders**gressions** and **learnin**ories**
- **make connections** for **planning intervention**

What is your personal learning intention?

Agenda

- Welcome
- Mathematics Task
- Evidence of Student Thinking
- Learning Progressions
- Learning Trajectories
- Intervention Strategies
- Intervention in Action
- Closing and Reflection

Mathematics Task

Planning the Task


JUICY DROP POP
JUICY DROP GUMMIES
SOUR PATCH KIDS

Our class bought a candy factory. Our factory produced 4824 bags of candy to sell! If we distribute the bags of candy equally to sell at 8 different stores, how many bags of candy will each store get?

Introducing the Task

Solve the Problem

_ _ _

$$4{,}824 \div 8$$

Anticipate Student Thinking

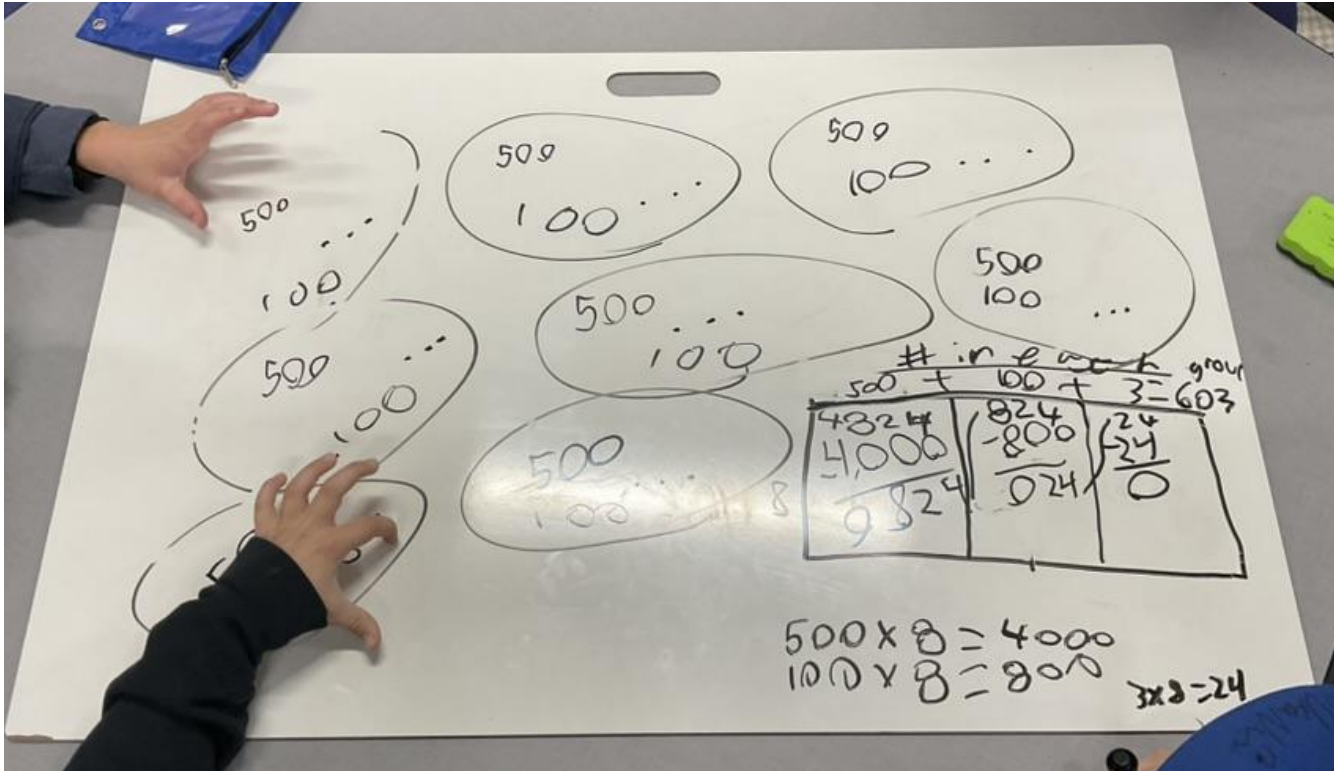
– – –

$$4{,}824 \div 8$$

- How might students solve the problem?
- What strategies and representations would they use?
- What errors or misconceptions might they have?

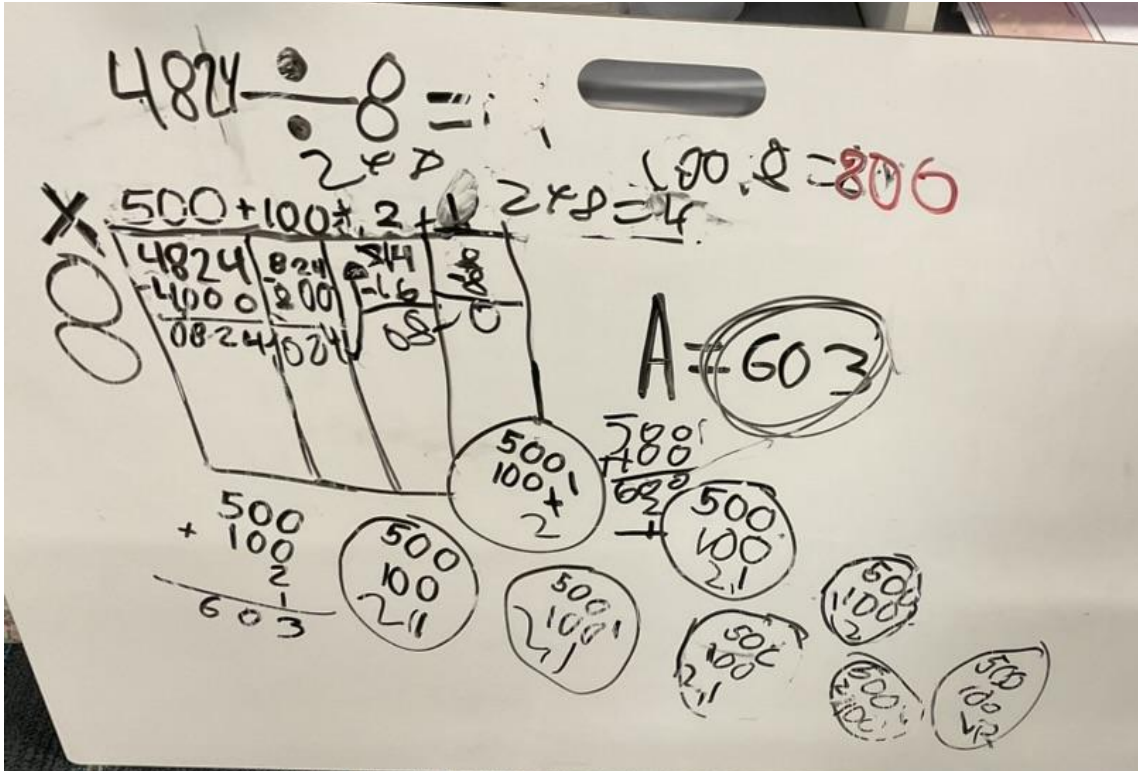
Evidence of Student Thinking

Group 1


500 ... 100
500 100 ...
500 100 ...
500 100 ...
500 100 ...
500 100 ...
500 100 ...
500 + 100 + 3 = 603
4824
-4,000
824
-800
024
24
-24
0
500 x 8 = 4000
100 x 8 = 800
3x8=24

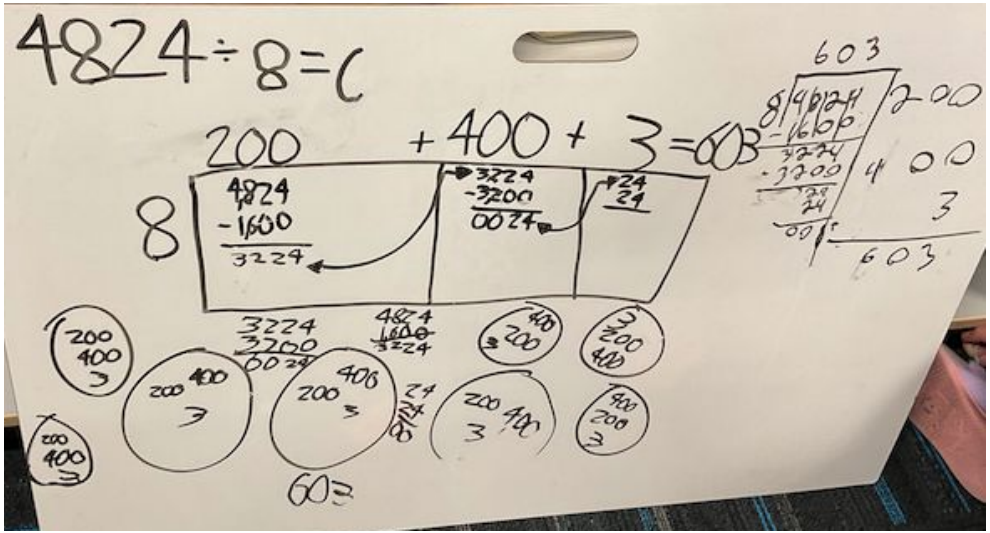
Group 1

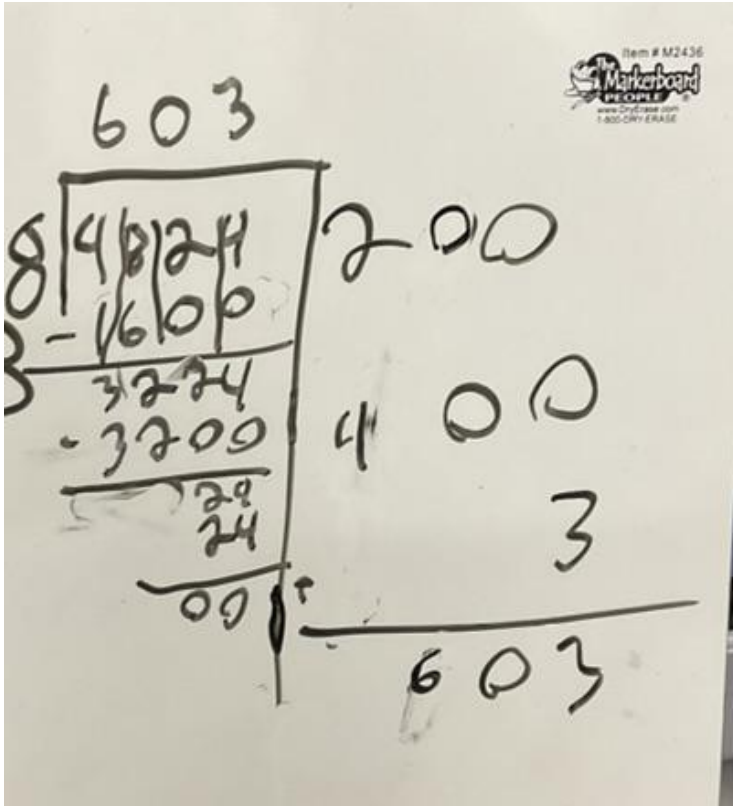
Group 2



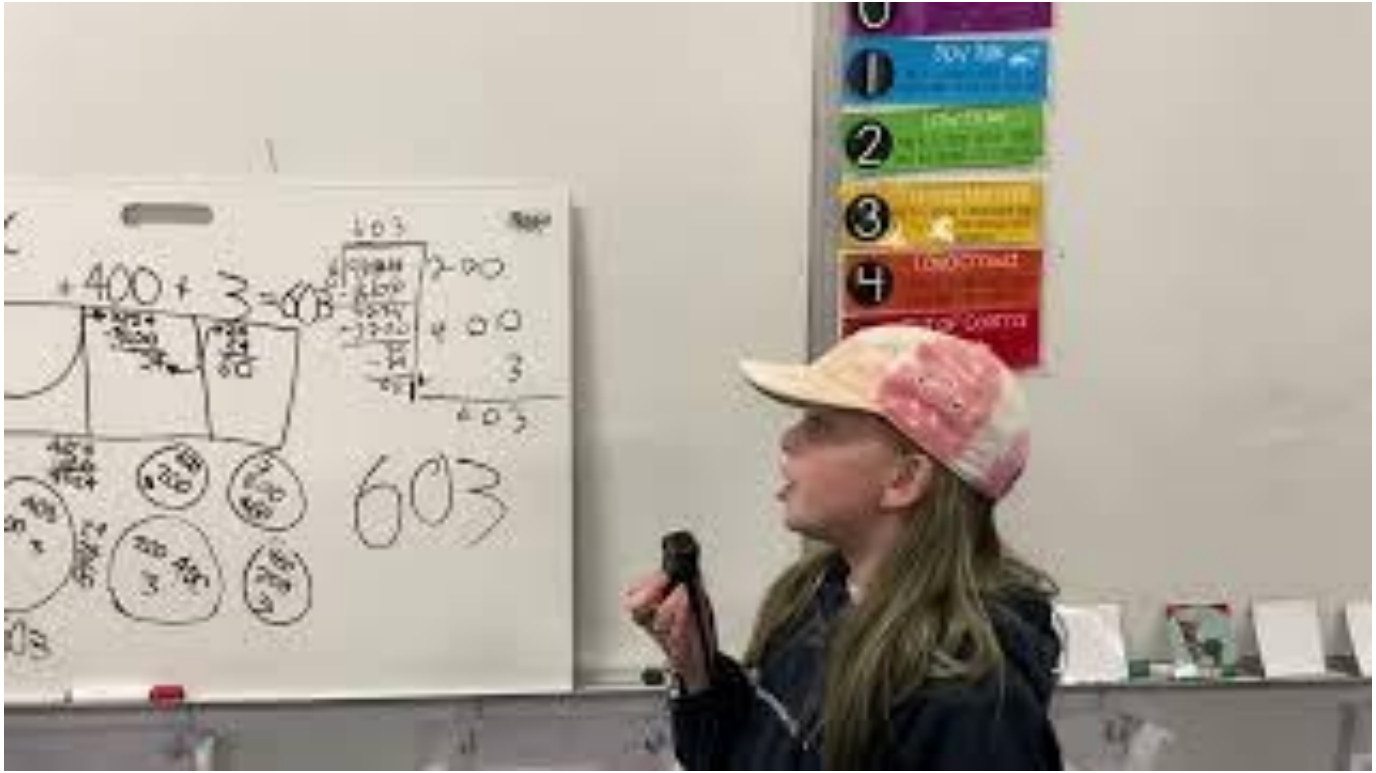
Group 2

Group 3


4824÷8=603
200 + 400 + 3 = 603
8
4824
-1600
3224
-3200
0024
24
603


603
8|4824
-1600
3224
-3200
24
200
400
3
603

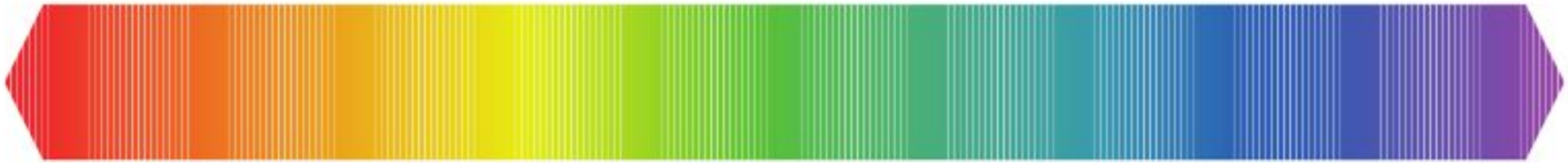
Group 3



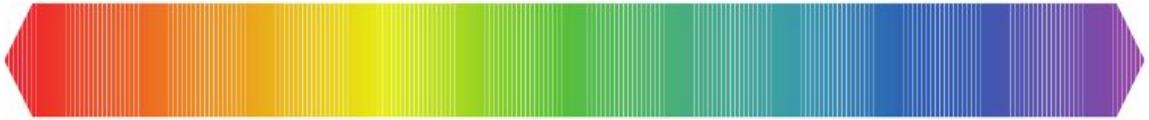
Learning Progressions & Learning Trajectories

Learning progressions and learning trajectories inform the pathways in which students learn content.

Sometimes the terms are used interchangeably, but depending on the source and context there are a few differences.



Continuums



"**Learning progressions, progress maps, developmental continuums,** and **learning trajectories** are all terms that have been used in the literature over the past decade. While many variations on the definition exist, the concept generally refers to **research-based, descriptive continuums of how students develop and demonstrate** deeper, broader, and more sophisticated **understanding over time.**" (Hess, 2014)

Building Blocks

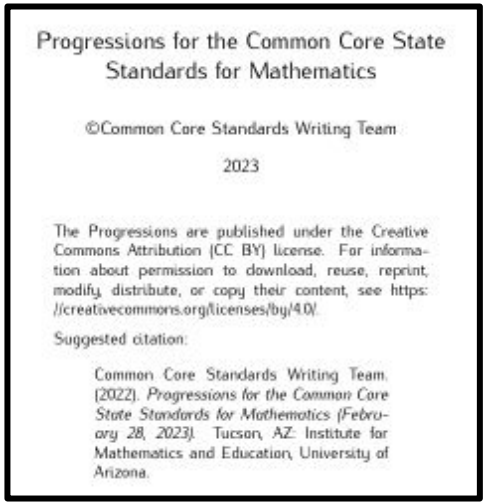
"Learning progressions describe **typical sequences of learning in specific areas** or disciplines. ... while learning standards describe what a student should have learned by a specific stage in their education, **learning progressions focus on the building blocks that contribute to mastering a particular skill**. This developmental approach maps the progress of a student through stages of increasing knowledge, skills, and understanding." (Kim & Scoular, 2017)

Standards Progressions

"The Standards in mathematics were built on **progressions of topics across a number of grade levels**, informed both by **research on children's cognitive development** and by the **logical structure of mathematics**."
(The University of Arizona, 2013)

Progressions Documents

Narrative documents describing the progression of a topic across a number of grade levels



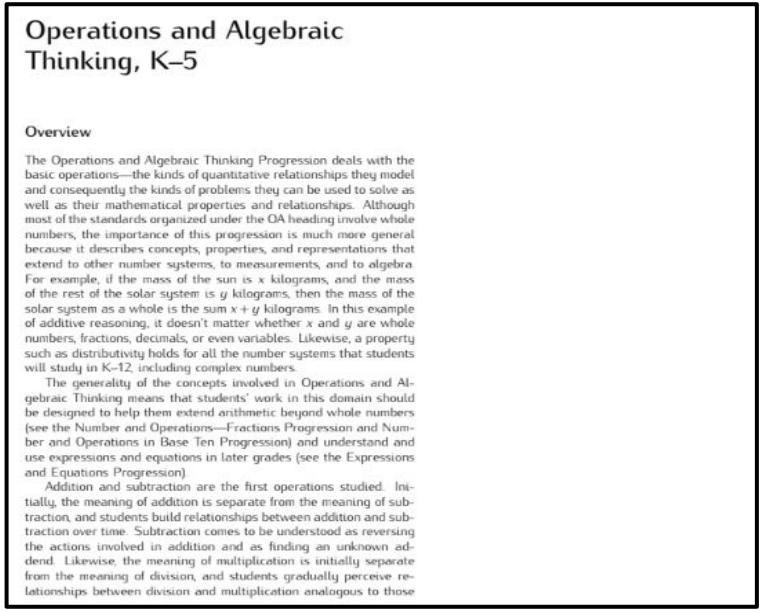
Progressions for the Common Core State Standards for Mathematics

Suggested citation:

Common Core Standards Writing Team. (2022). *Progressions for the Common Core State Standards for Mathematics (February 28, 2023)*. Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

https://mathematicalmusings.org/

Operations and Algebraic Thinking & Numbers and Operations in Base Ten



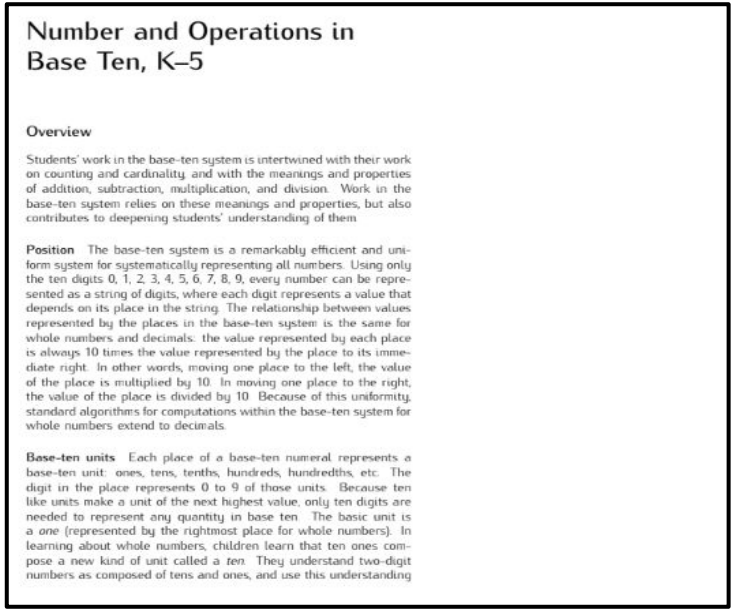
Operations and Algebraic Thinking, K–5

Overview

The Operations and Algebraic Thinking Progression deals with the basic operations—the kinds of quantitative relationships they model and consequently the kinds of problems they can be used to solve as well as their mathematical properties and relationships. Although most of the standards organized under the OA heading involve whole numbers, the importance of this progression is much more general because it describes concepts, properties, and representations that extend to other number systems, to measurements, and to algebra. For example, if the mass of the sun is x kilograms, and the mass of the rest of the solar system is y kilograms, then the mass of the solar system as a whole is the sum $x + y$ kilograms. In this example of additive reasoning, it doesn't matter whether x and y are whole numbers, fractions, decimals, or even variables. Likewise, a property such as distributivity holds for all the number systems that students will study in K–12, including complex numbers.

The generality of the concepts involved in Operations and Algebraic Thinking means that students' work in this domain should be designed to help them extend arithmetic beyond whole numbers (see the Number and Operations—Fractions Progression and Number and Operations in Base Ten Progression) and understand and use expressions and equations in later grades (see the Expressions and Equations Progression).

Addition and subtraction are the first operations studied. Initially, the meaning of addition is separate from the meaning of subtraction, and students build relationships between addition and subtraction over time. Subtraction comes to be understood as reversing the actions involved in addition and as finding an unknown addend. Likewise, the meaning of multiplication is initially separate from the meaning of division, and students gradually perceive relationships between division and multiplication analogous to those



Number and Operations in Base Ten, K–5

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

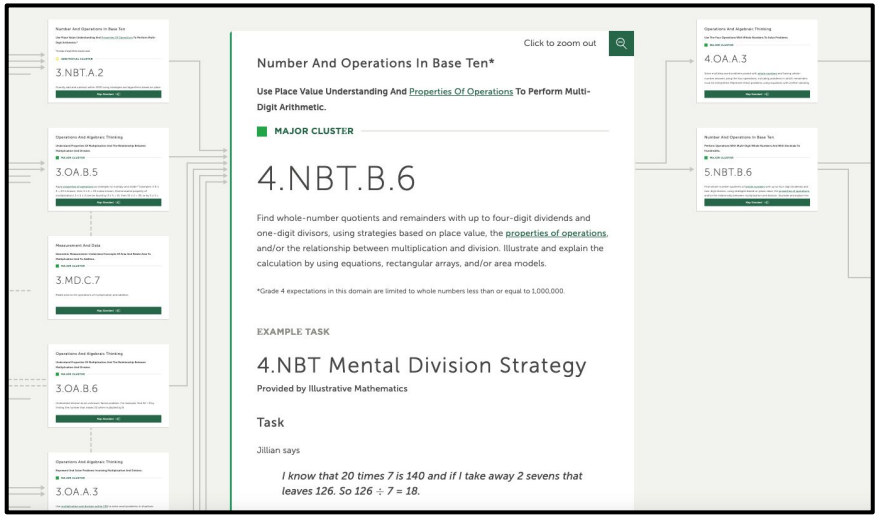
Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding

Coherence Map

Mathematics standards are not isolated concepts.

Standards relate to one another, both within and across grades. The Coherence Map illustrates the coherent structure that is fundamental to college- and career-ready standards.



https://achievethecore.org/coherence-map/

Standards Related to Multiplication and Division with Whole Numbers

Kindergarten

Count to tell the number of objects.

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

First Grade

Work with addition and subtraction equations.

1.OA.7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 − 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.

Second Grade

Work with equal groups of objects to gain foundations for multiplication.

2.OA.3 Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

2.OA.4 Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Reason with shapes and their attributes.

2.G.2 Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Number of Standards Per Grade Level

K	2
1	1
2	3
3	8
4	5
5	2
6	1

3-5 Standards Progression in Multiplication and Division

	Third Grade	Fourth Grade	Fifth Grade
Contexts/ Problem Types	Equal groups Array and area problems Measurement quantities • Product unknown • Group size unknown • Number of groups unknown	Comparison problems • Product unknown • Group size unknown • Multiplier unknown Interpret whole number remainders	All problem types from previous grade levels
Place Value	Multiply and divide within 100: • multiply one-digit by one-digit (e.g., 3 x 5) • multiply other problems within 100 (e.g., 2 x 48, 3 x 25) • Divide one and two-digits by one-digit without remainders Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 5 × 60, 9 × 80)	Multiply whole numbers: • two digits by one-digit • three digits by one-digit • four digits by one-digit • two two-digit numbers Divide whole numbers: • two-digits by one-digit • three-digits by one-digit • four-digits by one-digit	Multiply multi-digit whole numbers using the standard algorithm Divide whole numbers: • two-digits by two-digit • three-digits by two-digit • four-digits by two-digit
Fluency	Fluently multiply and divide within 100 using strategies Know from memory all products of two one-digit numbers	Continue to work on fluency expectations from Grade 3	Fluently multiply multi-digit whole numbers using the standard algorithm
Additional Concepts & Skills	Apply properties of operations (Commutative, Associative, Distributive) Solve two-step word problems using the four operations	Find factors, multiples, and prime and composite numbers Solve multistep word problems using the four operations	Write and interpret numerical expressions using the four operations Multiply and divide by powers of ten to understand the place value system

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HOW

How to Teach Topics

"Although standards are essential in identifying what topics to teach and when to teach them, they offer little insight into **how to teach those topics**. Learning trajectories (LTs) are **empirically grounded descriptions of how students' reasoning evolves from less to more sophisticated**. They can provide deep insight into **how to teach topics** during a single grade as well as **how topics develop and evolve across the grades.**" (Confrey, Shah, & Maloney, 2022).

Climbing Walls

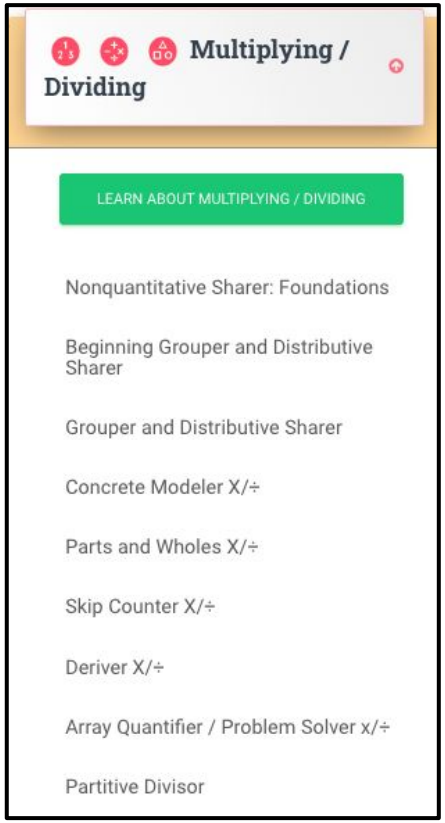
"We liken LTs to **a climbing wall rather than a ladder**. ... A ladder implies that students proceed uniformly through strictly prerequisite levels. **A climbing wall assumes climbers move upward from a variety of starting points through multiple paths.** LT levels, when envisioned as handholds, footholds, and obstacles, **make student thinking visible and predictable**, though probabilistic." (Confrey, Shah, & Maloney, 2022)

Natural Development

"Children follow **natural developmental progressions in learning**. Curriculum research has revealed **sequences of activities that are effective in guiding children through these levels of thinking.** These developmental paths are the basis for the learning trajectories. (Clements & Sarama, 2017/2019)

Learning and Teaching with Learning Trajectories

Differentiated instruction for early mathematics instruction is supported through understanding and use of Learning Trajectories - a goal, a developmental progression, and learning opportunities.



www.learningtrajectories.org/math/learning-trajectories/multiplying-dividing

Learning Trajectories in Multiplication and Division

Learning Trajectories	Numerical Parameters	Strategies and Representations
Nonquantitative Sharer: Foundations Beginning Grouper and Distributive Sharer Grouper and Distributive Sharer Concrete Modeler X/÷ Parts and Wholes X/÷ Skip Counter X/÷ Deriver X/÷ Array Quantifier / Problem Solver x/÷ Partitive Divisor See details at: https://www.learningtrajectories.org/math/learning-trajectories/multiplying-dividing	Multiply one-digit numbers: -Twos (doubles) -Fives (skip counting) -Tens (skip counting) -Zeros (use context) -Ones -Nines (derive from tens) -Threes -Fours (double the double) -Sixes -Sevens -Eights Multiply within 100 (e.g., 2 x 48) Divide within 100 without remainders (e.g., 60 ÷ 5, 64 ÷ 4) Multiply one-digit numbers by multiples of 10 (e.g., 5 × 60) Multiply whole numbers: -two digits by one-digit -three digits by one-digit -four digits by one-digit -two two-digit numbers Divide whole numbers: -two-digits by one-digit -three-digits by one-digit -four-digits by one-digit -interpret remainders Multiply multi-digit numbers using the standard algorithm Divide whole numbers: -two-digits by two-digit -three-digits by two-digit -four-digits by two-digit Divide multi-digit numbers using the standard algorithm	Multiplication and division by a single-digit multiplier/divisor: use physical objects and then visual drawings - Equal Groups - Arrays - Area Models - Number lines Multi-digit multiplication - Physical base-ten area models - Visual base-ten area models - Open arrays - Partial products - Algorithms Multi-digit division - Physical equal groups/ fair-share models - Physical and visual base-ten area models - Open arrays with written records - Repeated subtraction - Partial quotients - Algorithms Standard Algorithms: Prove that it produces a correct answer

Learning Progressions and Learning Trajectories

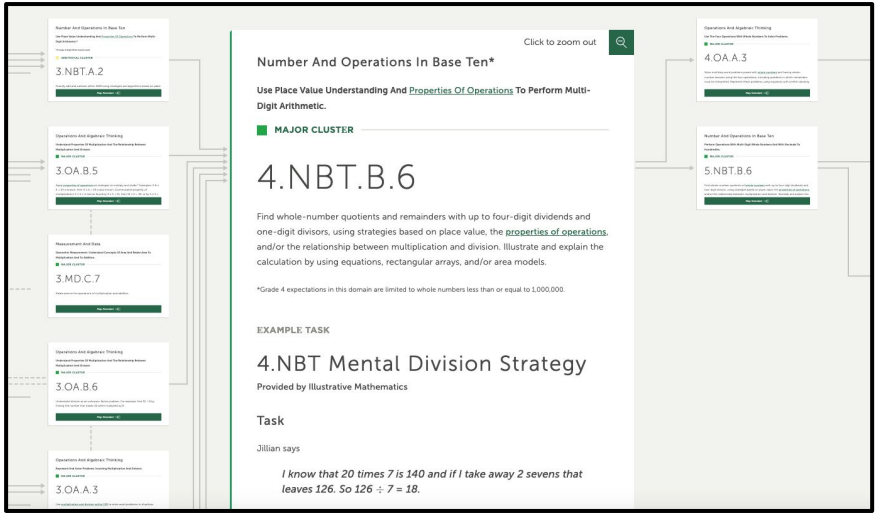
Definitions:

Learning progressions describe how big ideas advance across grade levels and the way standards connect over time. They tell us ***when*** students learn concepts.

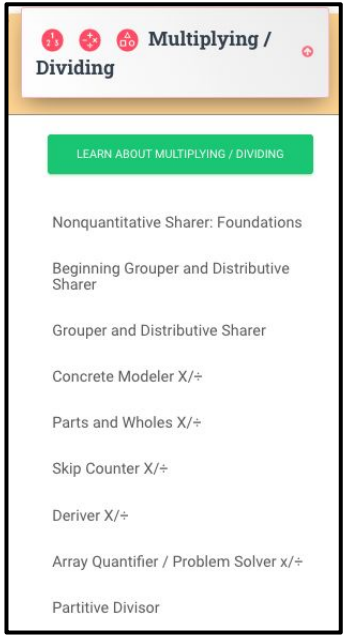
Learning trajectories refer to the pathways children naturally take to develop mathematical understanding. They tell us ***how*** students learn concepts.

Learning Progressions and Learning Trajectories

Examples:



https://achievethecore.org/coherence-map



https://www.learningtrajectories.org

Learning Progressions and Learning Trajectories

Applications:

- Planning for whole class instruction
- Planning for grade level instruction
- Knowing the flow of big ideas across time

- Planning for differentiated instruction
- Planning for skill-based instruction
- Knowing the flow of developing individual skills

Intervention Strategies

Intervention Strategies

- Learning Progressions
- Learning Trajectories
- Alignment of Intervention to Core Instruction
- Context
- Mathematical Language
- Representations

Reference:
Institute of Education Sciences: What Works Clearinghouse. (2021, March). WWC: *Assisting students struggling with mathematics: Intervention in the elementary grades.* Retrieved from https://ies.ed.gov/ncee/wwc/PracticeGuide/26.

Learning Progressions:

Use learning progressions to know when to teach concepts in relation to other concepts and grade level expectations.

Learning Trajectories:

Use learning trajectories to know how children develop understanding of concepts.

Alignment of Intervention to Core Instruction:

Systematically ensure that learning in intervention directly supports learning in Tier 1 instruction.

Context:

– – –

Present mathematical situations using words problems students can understand to deepen students' mathematical understanding and support their capacity to apply mathematical ideas.

Mathematical Language:

– – –

Allow students to naturally use informal language as they develop understanding of concepts. Once concepts are surfaced, teach clear and concise academic language.

Representations:

Provide students with physical tools to solve mathematical problems and connect those physical models to pictorial and other representations. Regularly include number lines as representations.

Intervention in Action

Planning for Intervention

Look For:

- Evidence of Student Thinking
- Evidence of Learning Progressions & Learning Trajectories
- Evidence of Intervention Strategies

Intervention on Using Basic Facts

Intervention with Context and Two-digit Dividends

There are 48 people going to the zoo.
4 people can fit in a car.
How many cars do we need?

Intervention with Context and Two-digit Dividends

What evidence did you see?

– – –

- Evidence of Student Thinking
- Evidence of Learning Progressions & Learning Trajectories
- Evidence of Intervention Strategies

Closing and Reflection

Thank you to Mrs. Snyder and her class!

Fourth Grade
Trailside Elementary
Vineyard, UT

What do you want to remember from the session?

Connect with Me

Thank You!

Connect with me

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Resources and References

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